SERIAL FACTORIAL DESIGNS

by

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Summary

The thesis is primarily concerned with change-over designs for t treatments, t^2 subjects and $2t$ periods such that each combination of treatments in any two consecutive periods occurs on exactly one subject and each treatment occurs twice with each subject. Such designs are here called Serial factorial change-over designs (SF designs).

These designs were introduced by Berenblut (1964) and further considered by Patterson (1970). The present thesis extends the methods of construction used by Patterson (1970) to give a wide range of SF designs within the general class.

When direct and residual effects are genuinely additive we show that all SF designs for a given t are equally suitable. Direct effects are always orthogonal to residual effects.

When direct and residual effects are not additive the designs differ in several respects but all have the property that direct effects are orthogonal to all direct × residual interaction components. The most important differences between designs are in respect of

(i) the orthogonality (or lack of orthogonality) between residual effects and direct × residual interactions.

(ii) the efficiency of estimation of interaction components.

We define a special subclass of designs, called R-orthogonal designs, with direct and residual effects orthogonal, not only to each other, but to all components of interaction. Another important subclass consists of those designs, called binary designs, in which no combination of treatments in two consecutive periods is repeated on the same subject. On the average, over all components, binary R-orthogonal designs maximise the efficiency of estimation of interaction.

Recommendations on choice of designs are given for the following

particular cases:

(a) the treatments are 3 or 4 equally spaced levels of a single quantitative factor.

(b) the treatments are 3 or 4 levels of a qualitative factor.

(c) the treatments are the 2×2 combinations of two factors each at two levels.

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Contents

Chapter One

Introduction

Designs in which several treatments are applied in successive periods to each subject (or experimental unit) in a cyclic sequence are known as changeover designs. These designs are frequently used if the subjects are very variable, expensive or scarce.

Changeover designs have been used in several fields of experimentation including

(i) nutrition experiments with dairy cattle

(ii) clinical trials in medical research

(iii) psychological experiments

(iv) long-term agricultural field experiments

(v) bioassay.

In this thesis, the same terms treatments, subjects and periods are used regardless of the field of application. Thus, the treatments in nutrition experiments may be the different amounts or several kinds of feeding stuffs; in long-term agricultural field experiments, the treatments may be the different amounts or several kinds of fertilisers. Subjects may be cows in dairy feeding trials and plots of land in agricultural field experiments. The duration of a period may be a week to ten days in nutrition experiments; a period in long-term agricultural field experiments may be as long as a year.

Changeover designs are generally used with the intention of obtaining high precision in estimation of treatment comparisons; this is possible because differences between subjects are eliminated from experimental error. However, this advantage may be obtained only at the expense of possible complications arising from the after-effects of a treatment when its application is discontinued: any treatment comparison may then be influenced not only by the <u>direct effects</u> of the treatments concerned but also by previous treatments or treatment effects. These after-effects are known as <u>residual effects</u>.

There are several approaches to this problem of residual effects. These include the following:

(a) the residual effects are assumed to be non-existent or negligible.

(b) a rest period (with no treatment applied) can sometimes be inserted between the treatment periods in order to eliminate the residual effects of the previous treatment.

(c) Special types of changeover designs are used that allow for the separate estimation of both direct and residual effects.

In this thesis, we are only concerned with the approach through special types of changeover designs.

Residual effects of a treatment may occur in any period of the experiment after the application of the treatment. To help distinguish between residual effects of a treatment applied in, say, period i, the residual effect of that treatment in period $i+1$ is called first residual effect, the residual effect in period $i+2$ is called second residual effect and so on. In many fields of work, the first residual effects, if any, will usually be largest; the second and higher-order residual effects will usually be progressively smaller and can often be reasonably assumed to be negligible. Therefore, throughout the remainder of this thesis, we refer to first residual effects simply as residual effects.

Many changeover designs have been constructed to facilitate the separate estimation of direct effects and (first) residual effects. These include designs constructed by

(1) Cochran et al (1941)

(2) Williams (1949)

(3) Lucas (1957)

(4) Patterson (1951)

(5) Patterson (1952)

(6) Patterson and Lucas (1959)

(7) Davis and Hall (1969)

(8) Federer and Atkinson (1964)

(9) Berenblut (1964)

(10) Berenblut (1967b, 1968).

The above designs will be examined in greater details in Chapter Two. (Williams (1950) constructed designs that also provide for the estimation of second residual effects). These designs are only suitable for stable conditions. By stable conditions we mean that conditions that may influence the results of an experiment do not change from one period to the next. It can often be reasonably assumed that conditions are stable in nutrition experiments and clinical trials. But in long-term agricultural field experiments, the conditions from one period to the next are often unstable, in which case designs for unstable conditions have to be used. However, in this thesis, we are only concerned with designs for stable conditions.

Further complexity arises if not only the (first) residual effect exists in a period but it interacts with the direct effect of a treatment applied in that period. Such interaction of direct effects and (first) residual effects will be known as direct × first residual interaction. For brevity, throughout this thesis, we refer to direct × first residual interaction simply as direct × residual interaction.

Much of the earlier literature concentrated on the construction of designs in which

(1) each treatment level is followed in the next period by each other level equally often

(2) each treatment level occurs exactly once with each subject.

Condition (1) gives balance, but not orthogonality, between direct effects and (first) residual effects. Condition (2) gives balance with respect to subjects. Sometimes these conditions impose limitations on the availability of designs. Thus, for example, condition (2) requires that the number of treatments is the same as the number of periods. Patterson (1951, 1952), Patterson and Lucas (1962) and Davis and Hall (1969) obtain full or partial balance with respect to subjects with the number of periods smaller than the number of treatments. In their designs condition (2) is replaced by (2') in which each level occurs at most once with each subject.

Lucas (1957) and Patterson and Lucas (1959) showed that orthogonality between direct effects and (first) residual effects can be obtained by repeating in an extra period the treatment levels of the final period of a design which otherwise satisfies conditions (1) and (2) or (1) and (2').

Berenblut (1964) achieved orthogonality of direct and (first) residual effects by an entirely different method. His designs require t^2 subjects and 2t periods for t treatments. In his designs, the essential conditions are

(3) each combination of treatment levels in any two consecutive periods occurs on exactly one subject,

(4) each treatment level occurs twice with each subject.

Berenblut (1964) gave the details of construction of his designs and later (Berenblut, 1967a) described the analysis. He also pointed out, without giving details, that the designs have an additional property in that all components of direct × residual interaction are orthogonal to direct and residual effects (Berenblut, 1968).

Patterson (1970) constructed a wider class of designs for t treatments, t^2 subjects and 2t periods. These designs also satisfy conditions (3) and (4) given above. The designs include those constructed by Berenblut (1964) and have the same properties as those of Berenblut (1964) given above.

Although Patterson's method is general, he was particularly concerned with designs for four treatments. Also he was interested in the case in which the treatments are the four equally spaced levels of a single quantitative factor. He showed that, although all designs for four treatments are equally efficient in estimating direct and residual effects, some designs are more efficient than others in the estimation of linear direct × linear residual interaction and may therefore be preferred.

The present thesis deals with the construction, properties and analysis of a general class of designs for t treatments, t^2 subjects and $2t$ periods satisfying conditions (3) and (4) given above with reference to Berenblut (1964). Condition (3) is a property of serial factorial designs (Patterson, 1968). However, not all types of serial factorial designs are suitable for changeover trials. We will, therefore, refer to the particular designs considered here as Serial factorial changeover designs (SF designs for short).

Chapter Two deals with designs for the estimation of additive direct and residual effects. These designs include conventional changeover designs satisfying conditions (1) and (2), (1) and (2') and extra-period designs, as well as the general class of SF designs.

In Chapter Three, we will consider properties of orthogonality, balance and efficiency in SF designs with particular reference to the estimation of direct × residual interaction. Chapter Four deals with methods of construction and Chapter Five establishes guidelines for choosing between the designs in various practical situations.

Chapter Two

Designs for model with additive direct effects and residual effects

2.1 Introduction.

In this chapter, we first review the existing change-over designs. A new class of designs is considered. We then consider a general model for change-over designs with provisions for only direct and residual effects. (This model will be extended in Chapter Three to provide for direct × residual interaction). Using this model, the existing change-over designs will be assessed on their properties of orthogonality, balance and efficiency.

2.2 Review of existing change-over designs.

In this section, we review the existing change-over designs, These designs also serve as illustrations for the model considered in the later section.

2.2.1 Ordinary Latin squares.

A single ordinary Latin square may be used as a change-over design. As an example, a design is given in Table 2.1. This design is not suitable when there are residual effects because it is not balanced for direct effects. A design is said to be balanced for treatment (direct or residual) effects if all treatment differences have the same variance. (A formal definition of balance will be given in section 2.7). Similarly, the design is not balanced for residual effects.

Table 2.1 Ordinary Latin square design .

	Subject			
Period	I	II	III	IV
1	1	2	3	4
2	2	1	4	3
3	3	4	2	1
4	4	3	1	2

2.2.2 Design by Cochran et al (1941).

To overcome this problem of balance when residual effects exist, Cochran et al (1941) constructed change-over designs using orthogonal Latin squares. (See Fisher and Yates (1963)). The columns of these orthogonal Latin squares are regarded as subjects and the rows as periods. The designs require t periods and $t(t-1)$ subjects for t treatments.

An essential feature of these designs is that each treatment is immediately followed in the next period by every other treatment (but not itself) equally often over the whole design and also between any two consecutive periods. The designs are balanced for direct effects and also for residual effects. They are also balanced for direct effects and residual effects when $p < t$ periods are considered. Note that the second residual effects are also balanced in the designs since each treatment is followed 2 periods later by every other treatment. A design for 3 treatments is given in Table 2.2.

Table 2.2 Design by Cochran et al (1941) for 3 treatments.

	Subject					
Period	I	II	III	IV	V	VI
1	1	2	3	1	2	3
2	2	3	1	3	1	2
3	3	1	2	2	3	1

Blocking of subjects is possible in the designs without confounding direct and residual effects with blocks. Each Latin square can form a block.

2.2.3 Design by Williams (1949).

Williams (1949) constructed designs which retain this property of balance for direct and residual effects but require fewer number of subjects than the designs by Cochran et al (1941). Special types of Latin squares are used. For even number of treatments, the design consists of any simple Latin square such that each treatment is immediately succeeded by every other treatment exactly once. For odd number of treatments, the design consists of any two Latin squares such that each treatment is immediately succeeded by every other treatment exactly twice. As an example, the design for 4 treatments is given in Table 2.3.

Table 2.3 Williams' design (1949) for 4 treatments.

	Subject			
Period	I	II	III	IV
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

The advantage of these designs is that they require fewer number of subjects. But the disadvantage is that the designs are not balanced for direct and residual effects when $p < t$ periods are considered. Also the designs deal equally well with first residual effects but not second or higher residual effects.

When two Latin squares (each balanced by itself) are used for the design with 4 treatments, each Latin square can form a block. An alternative method of blocking is available. It is shown in Table 2.4. This method is less efficient if there is block $\times$ direct effect or block $\times$ residual effect interaction.

Table 2.4 An alternative method of blocking of Williams' design (1949) with four treatments and eight subjects.

	Block 1				Block 2			
	Subject							
Period	I	II	III	IV	V	VI	VII	VIII
1	1	2	3	4	1	2	3	4
2	2	1	4	3	3	4	1	2
3	3	4	1	2	2	1	4	3
4	4	3	2	1	4	3	2	1

2.2.4 Designs by Lucas (1957).

For the designs considered so far, direct effects are not orthogonal to residual effects and therefore the estimates of direct effects have to be adjusted for residual effects and vice versa. (A formal definition of orthogonality will be given in section 2.5). Lucas (1957) constructed designs which give orthogonality of direct and residual effects. They are obtained by repeating in an extra period the treatments in the last period of designs by Cochran et al (1941) and Williams (1949). Earlier, Patterson (1951) suggested this method but did not elaborate on it. There are, however, other ways of getting orthogonality. (See subsection 2.2.9).

The essential feature of these designs is that each treatment is immediately followed by every other treatment and itself equally often. In contrast to designs by Cochran et al (1941) and Williams (1949), residual effects are orthogonal to subject differences. However, in common with these designs, direct effects are not orthogonal to subjects when residual effects are present.

2.2.5 Designs by Patterson (1951, 1952).

In the designs considered so far, the number of periods, p, is completely determined by the number of treatments, t, and also $p \geq t$.

Patterson (1951, 1952) gave designs in which the number of periods is fewer than the number of treatments but the designs are still balanced for direct and residual effects. These designs are obtained by several methods of construction. These include the method of differences, the use of complete sets of orthogonal Latin squares and the use of cyclic incomplete block designs. Blocks in cyclic incomplete block designs are regarded as subjects, and rows constitute periods. A design obtained from the use of cyclic incomplete block design is given Table 2.5.

Table 2.5 A design by Patterson (1951, 1952) for seven treatments, fourteen subjects and four periods.

	Subject													
Period	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII	XIV
1	1	2	3	4	5	6	7	1	2	3	4	5	6	7
2	2	3	4	5	6	7	1	7	1	2	3	4	5	6
3	4	5	6	7	1	2	3	5	6	7	1	2	3	4
4	7	1	2	3	4	5	6	2	3	4	5	6	7	1

An essential feature of these designs is that each treatment is immediately succeeded in the next period by every other treatment. The disadvantage of these designs is that direct and residual effects are estimated with lesser precision.

Patterson (1952) gave conditions for balance in the change-over designs in which no treatment is applied to any subject in more than one period. (Such designs satisfying the conditions for balance are known as basic change-over designs). Patterson and Lucas (1959) gave more general conditions for balance in that the number of treatments is not restricted to be equal to the number of subjects in a block. Therefore, these general conditions are given.

If there are t treatments and p periods and the subjects are arranged in b blocks of k subjects, where $k>2$, the conditions, which must be satisfied for all pairs of treatments i and j, $i \neq j$, $i,j=1, 2, \ldots, t$, are as follows:

1. Treatment i occurs in each period in bk/t blocks.
2. Treatment j also occurs in each period in $bk(k-1)/t(t-1)$ of these blocks.
3. $\lambda = bkp(p-1)/t(t-1)$ of the bkp/t subjects which receive treatment i in some period receive treatment j in another period.
4. λ/p subjects receive treatment i then treatment j in successive periods.
5. λ/p subjects receive treatment i in the p^{th} period and treatment j in another period.

<u>2.2.6 Designs by Patterson and Lucas (1959).</u>

By repeating in an extra period the treatments in the last period of designs by Patterson (1951, 1952), Patterson and Lucas (1959) obtained designs in which direct effects are orthogonal to residual effects. These designs require fewer number of periods than designs by Lucas (1957) but direct and residual effects are estimated with lesser precision.

<u>2.2.7 Designs by Davis and Hall (1969).</u>

Davis and Hall (1969) constructed a class of cyclic change-over (CCO) designs. These CCO designs exist for any number of treatments and periods. They are defined as a simple extension of the cyclic incomplete block (CIB) designs: the blocks of CIB designs are regarded as subjects, and the rows constitute periods.

In these designs, all direct effects or residual effects are not necessarily estimated with equal precision. Therefore, in using these cyclic designs, it is important to ensure that the degree of imbalance is of no practical importance. A design that is not balanced for direct and residual effects is given in Table 2.6. In this design, each treatment is not immediately succeeded in the next period by every other treatment; for example, treatment 1 is never succeeded immediately by treatment 5. However, the advantage of these CCO designs is their greater availability. Also these designs require, in general, fewer subjects.

Table 2.6 A design by Davis and Hall (1969) for six treatments, twelve subjects and three periods.

	Subject											
Period	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII
1	1	2	3	4	5	6	1	2	3	4	5	6
2	4	5	6	1	2	3	6	1	2	3	4	5
3	5	6	1	2	3	4	2	3	4	5	6	1

2.2.8 Designs by Federer and Atkinson (1964).

Federer and Atkinson (1964) constructed a class of designs for t treatments requiring $(qt+1)$ periods and $st(t-1)$ subjects, for integers s and q. These designs are constructed using orthogonal Latin squares. The essential features of these designs are that each treatment is immediately followed by every other treatment equally often over the whole design and between any two consecutive periods, the treatments in periods 1 to t correspond to the treatments of one or more set of $(t-1)$ orthogonal $t \times t$ Latin squares, the treatments in periods $t+1$ to $2t$ correspond to the treatments of the same number of sets of $(t-1)$ orthogonal $t \times t$ Latin squares, ..., etc. The general method of construction of these designs is apparent from the design for three treatments, six subjects and seven periods given in Table 2.7. This design is also given by Federer (1955).

Table 2.7 Design by Federer and Atkinson (1964) for 3 treatments, 6 subjects and 7 periods.

	Subject					
Period	I	II	III	IV	V	VI
1	1	2	3	1	2	3
2	2	3	1	3	1	2
3	3	1	2	2	3	1
4	1	2	3	1	2	3
5	3	1	2	2	3	1
6	2	3	1	3	1	2
7	1	2	3	1	2	3

These designs are balanced for direct effects and residual effects. The variances of direct effects and residual effects are also more nearly equal. The relative variances of direct effects and residual effects approach equality as the number of periods increases.

2.2.9 Designs by Berenblut (1964).

Berenblut (1964) gave a method of construction of designs, for t treatments requiring t^2 subjects and $2t$ periods, in which direct effects are orthogonal to residual effects. (An analysis of these designs is given in Berenblut (1967a)). This does not define a complete class of designs. A suggested definition is that all t^2 combinations of treatment levels occur in any two consecutive periods and each treatment occurs twice on each subject. Therefore, the designs have a factorial structure and can be regarded as serial factorial designs. (The present thesis deals with these serial factorial designs.)

In these designs, direct effects are also orthogonal to subjects.

Therefore, direct effects are estimated with full efficiency. However, the disadvantage is the larger numbers of periods and subjects required. A design by Berenblut (1964) for three treatments is given in Table 2.8.

Table 2.8 Berenblut's design (1964) for 3 treatments, 9 subjects and 6 periods.

	Subject								
Period	I	II	III	IV	V	VI	VII	VIII	IX
1	1	2	3	1	2	3	1	2	3
2	1	2	3	2	3	1	3	1	2
3	2	3	1	2	3	1	2	3	1
4	2	3	1	3	1	2	1	2	3
5	3	1	2	3	1	2	3	1	2
6	3	1	2	1	2	3	2	3	1

2.2.10 Designs by Berenblut (1967b, 1968).

Berenblut (1967b, 1968) constructed designs for four and five treatments by using two special Latin squares for each design. These are very special cases of designs with serial factorial properties but fewer numbers of periods and subjects than in designs by Berenblut (1964). In these designs, the linear residual effect is orthogonal to all components of direct effects. Only one design for four treatments exists. It is given in Table 2.9.

Table 2.9 Design by Berenblut (1967b, 1968) for four treatments.

	Subject							
Period	I	II	III	IV	V	VI	VII	VIII
1	1	2	3	4	1	2	3	4
2	2	1	4	3	3	4	1	2
3	4	3	2	1	4	3	2	1
4	3	4	1	2	2	1	4	3

2.2.11 Stability of conditions.

Inherent in the analysis of change-over designs given by most authors is the assumption of stability of conditions. By this we mean that the conditions in the experiment do not change from one period to the next, that is, there is no treatment × period interaction. (However, Patterson (1950) and Lucas (1951) showed that different relative amounts of information on direct effects and residual effects are contained in linear, quadratic, ..., etc., components of time trends).

Designs by Patterson (1951, 1952) and Patterson and Lucas (1959) are not suitable for unstable conditions. In general, Latin square designs should also be restricted to stable conditions. But, some designs are much more sensitive to unstable conditions than others. These include designs by Williams (1949) and the extra-period designs by Lucas (1957) where the last two periods may have special conditions, for example, the end of lactation cycle in dairy cows. (See Lucas (1960)). Designs using orthogonal Latin squares are better than others when partial analysis is required by omitting some of the periods ~~because periods can be omitted for analysis~~. However, the very best design for unstable conditions are serial factorial designs including designs by Berenblut (1964) and Patterson (1968).

2.2.12 Choice of a design.

The choice of a design for an experiment depends very much on the experimental situation and the aim of the experiment. In a nutrition experiment on dairy cows, the number of periods is often restricted to four or five. This would, in effect, rule out the use of such designs as those of Berenblut (1964) even though these designs have better orthogonality properties than the other designs.

Again, in a clinical trial, the number of subjects is restricted. Designs requiring larger number of subjects may, therefore, be ruled out. However, in long-term agricultural field experiments, there are fewer restrictions on the numbers of subjects and periods. Designs by Berenblut (1964) may, therefore, be useful in this application, particularly as they are less sensitive than other designs to unstable conditions.

Furthermore, if the experimenter anticipates the existence of (first) residual effects but is not interested in estimating them, then designs using complete Latin squares, except the design in Table 2.1 , seem preferable to the extra-period designs. However, if large residual effects seem likely to exist and interest lies in estimating them, besides direct effects, the extra-period designs including those of Lucas (1957) may be preferable. Of course, if there are no restrictions on the numbers of subjects and periods, designs by Berenblut (1964) are the most suitable.

2.3 New A-class designs.

In this section, a class of designs for t treatments, $(qt+1)$ periods and $st(t-1)$ subjects, for integers q and s , is obtained from a modification of the designs by Federer and Atkinson (1964). (See subsection 2.2.8). These designs will be known as A-class designs. For designs with t treatments and $t(t-1)$ subjects, the treatments in the first t periods correspond to the treatments of the $(t-1)$ orthogonal $t \times t$ Latin squares. If only $(t+1)$ periods are required, the treatments in period t are repeated in period $(t+1)$. (These designs correspond to designs by Lucas (1957) in subsection 2.2.4). If $(2t+1)$ periods are required, the treatments in periods $(t+1)$ to $2t$ of each subject are obtained by reversing the order of treatments in the previous t periods, in this case, periods 1 to t: that is, the treatment in period $(t+1)$ is the same as the treatment in period t, the treatment in period $(t+2)$ is the same as the treatment in period $(t-1)$, ..., etc. Then the treatments in period $(2t+1)$ are obtained by repeating the treatments in period $2t$. The same procedure is applied to obtain the treatments for the other sets of $t(t-1)$ subjects. Thus, the general design is obtained. A design for three treatments, six subjects and seven periods is given in Table 2.10.

Table 2.10 A new A-class design for three treatments, six subjects and seven periods.

	Subject					
Period	I	II	III	IV	V	VI
1	1	2	3	1	2	3
2	2	3	1	3	1	2
3	3	1	2	2	3	1
4	3	1	2	2	3	1
5	2	3	1	3	1	2
6	1	2	3	1	2	3
7	1	2	3	1	2	3

The designs are such that each treatment is immediately followed in the next period by every other treatment, including itself, equally often. We will show in section 2.5 that direct effects are orthogonal to residual effects, and residual effects are orthogonal to subject differences. Thus, residual effects are estimated with full efficiency. This class of designs will be compared with designs by Federer and Atkinson (1964) and Berenblut (1964) in section 2.9 .

2.4 A model for changeover design.

We now consider a general model with provision for estimating direct and first residual effects but ignoring the blocking of subjects. It can be expressed as

$$\underset{(n\times 1)}{y_1} = 1_{(n)}\, \mu_1 + X_1\, \tau_1 + I_{(n)}\, \beta + \epsilon_1 \,, \tag{2.1}$$

$$\underset{(n\times 1)}{y_i} = 1_{(n)} \underset{(1\times 1)}{\mu_i} + \underset{(n\times t)}{X_i} \underset{(t\times 1)}{\tau_i} + \underset{(n\times t)}{X_{i-1}} \underset{(t\times 1)}{\rho_i} + I_{(n)} \underset{(n\times 1)}{\beta} + \underset{(n\times 1)}{\epsilon_i} \qquad i = 2, 3, \ldots, p.$$

The notation is as follows:

t is the number of treatments in each period,

n is the number of subjects,

p is the number of periods,

y_i is a vector of observations in period i ,

subjects are in standard order, and elements of y_i and β both follow this order for all i .

$I_{(n)}$ and $1_{(n)}$ are defined in Appendix A at the end of the thesis. Treatment effects and subject effects are regarded as fixed. τ_i, ρ_i, and β are the vectors of direct, residual and subject effects respectively in period i with the restrictions

$$1_{(t)}^T \tau_i = 1_{(t)}^T \rho_i = 0 \quad \text{for all } i \,,$$

$$\text{and} \quad 1_{(n)}^T \beta = 0 \,.$$

X_i, X_{i-1} are the first-order incidence matrices defined below. (Second-order incidence matrices will be defined in Chapter Three. Throughout the remainder of Chapter Two, however, we refer to first-order incidence matrices simply as incidence matrices).

If each subject has just one treatment in each period, then

$$X_i 1_{(t)} = 1_{(n)}, \quad i = 1, 2, \ldots, p \qquad (2.2)$$

where cell (u, v) of $X_i = 1$ if subject u receives treatment v in period i,

$= 0$ otherwise.

For example, the design by Cochran et al in Table 2.2 has the following incidence matrices.

$$X_1 = \begin{array}{c|c} & \text{Treatment} \\ & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{cc} & 1 \\ & 2 \\ \textbf{subject} & 3 \\ & 4 \\ & 5 \\ & 6 \end{array} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \qquad X_2 = \begin{array}{c|c} & \text{Treatment} \\ & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{cc} & 1 \\ & 2 \\ \textbf{subject} & 3 \\ & 4 \\ & 5 \\ & 6 \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

The model (2.1) differs from the models given by other authors on changeover designs in that a separate equation is given for each period. Given a suitable design, this allows the estimation of separate τ_i and ρ_i for different periods. However, in common with most authors, we will be mainly concerned with the special case where

$$\tau_i = \tau \quad \text{for all } i,$$

$$\rho_i = \rho \quad \text{for all } i.$$

Therefore the restrictions on τ_i's and ρ_i's are reduced to

$$1_{(t)}^T \tau = 0 .$$

$$1_{(t)}^T \rho = 0 .$$

2.4.1 Error models.

Let ϵ be the $np\times1$ vector such that

$$\epsilon = \begin{bmatrix} \epsilon_1 \\ \vdots \\ \epsilon_i \\ \vdots \\ \epsilon_p \end{bmatrix}$$

where ϵ_i is the $n\times1$ vector defined in model (2.1). We also define scalars ϵ_{ij}'s such that

$$\epsilon_i = \begin{bmatrix} \epsilon_{i1} \\ \vdots \\ \epsilon_{ij} \\ \vdots \\ \epsilon_{in} \end{bmatrix}$$

We assume that the errors in different subjects are uncorrelated, that is,

$$E(\epsilon_{ij}\ \epsilon_{i'j'}) = 0 \qquad j \neq j' \text{ , any } i, i' \text{ .}$$

We also assume the same error model for every subject. Then the general error model can be written

$$E(\epsilon) = 0$$

$$E(\epsilon\epsilon^T) = (\underset{(p\times p)}{\Sigma} \otimes I_{(n)})\ \sigma^2$$

(The (right) direct product of matrices is defined in Appendix B at the end of the thesis).

Various particular error models can be envisaged. (See Finney (1956)). We shall consider the cases when Σ takes one of the following three forms.

1. $$\underset{(p\times p)}{\Sigma} = \begin{bmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{p-1} \\ \rho & 1 & \rho & \cdots & \rho^{p-2} \\ \rho^2 & & 1 & & \\ \vdots & & & \ddots & \\ \rho^{p-1} & & & & 1 \end{bmatrix}$$

2. $$\underset{(p\times p)}{\Sigma} = \begin{bmatrix} 1 & \rho & 0 & \cdots & & 0 \\ \rho & 1 & \rho & 0 & \cdots & 0 \\ 0 & & 1 & & & \\ \vdots & & & & & \\ 0 & 0 & 0 & & \rho & 1 \end{bmatrix}$$

3. $$\underset{(p\times p)}{\Sigma} = I_{(p)}$$

Σ takes the first form when we assume that the correlation of the errors is accounted for by the first-order autoregressive specification

$$\epsilon_{ij} = \rho\,\epsilon_{i-1,\,j} + u_{ij} \quad \text{for subject } j$$

where the u_{ij}'s are independent, identically distributed random variables with $Eu_{ij} = 0$, $Eu^2_{ij} = \sigma_u^2$ and $|\rho| < 1$. (See Goldberger (1964)). Although such an error model is frequently used in economic time series, there may be occasions when it holds in changeover trials.

The second form of Σ is a particular case of the first form when we assume that powers of ρ are zero, that is, $\rho^2 = \rho^3 = \ldots = \rho^{p-1} = 0$. In this case, we assume a constant correlation between the errors in any two consecutive periods but no correlation of errors for periods more than one apart.

Σ takes the third form when we assume that the errors between any two periods are uncorrelated. In common with most authors on changeover designs, we assume $\Sigma = I_{(p)}$ for our analysis.

2.4.2 Normal equations.

The normal equations for estimating τ, ρ and β are

$$\sum_1 X_i^T y_i = \sum_1 X_i^T 1_{(n)}\, \mu_i + \sum_1 X_i^T X_i\, \hat{\tau} + \sum_2 X_i^T X_{i-1}\, \hat{\rho} + \sum_1 X_i\, \hat{\beta} \qquad (2.3)$$

$$\sum_2 X_{i-1}^T y_i = \sum_2 X_{i-1}^T 1_{(n)} \mu_i + \sum_2 X_{i-1}^T X_i \hat{\tau} + \sum_2 X_{i-1}^T X_{i-1} \hat{\rho} + \sum_2 X_{i-1} \hat{\beta} \quad (2.4)$$

$$\sum_1 y_i = \sum_1 1_{(n)} \mu_i + \sum_1 X_i \hat{\tau} + \sum_2 X_{i-1} \hat{\rho} + p\,\hat{\beta} \quad (2.5)$$

where $\sum_1 = \sum_{i=1}^{p}$, $\sum_2 = \sum_{i=2}^{p}$.

We will call $X_{i-1}^T X_i$ <u>the coincidence matrix</u> with cell (u, v) equal to the number of subjects with treatment u in period (i-1) and treatment v in period i .

Note $X_{i-1}^T X_i = 0$ for $i = 1$.

The sum of the coincidence matrices, $\sum_2 X_{i-1}^T X_i$, has cell (u, v) equal to the number of subjects on which treatment u is followed in the next period by treatment v . This is the matrix appearing in the Normal equations (2.3) and (2.4) for the special model we are concerned with here. For example, for the design by Cochran et al (1941) in Table 2.2

$$X_1^T X_2 = \begin{array}{ll} & \text{treatment in period 2} \\ & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{lr} \text{treatment} & 1 \\ \text{in period 1} & 2 \\ & 3 \end{array} & \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array}$$

$$\sum_2 X_{i-1}^T X_i = \begin{array}{ll} & \text{treatment in present period} \\ & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{lr} \text{treatment} & 1 \\ \text{in previous} & 2 \\ \text{period} & 3 \end{array} & \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix} \end{array}$$

If each treatment occurs $^n/_t$ times in each period, where $^n/_t$ is an integer, then the column sums of X_i are all equal to $^n/_t$, that is,

$$1_{(n)}^T X_i = \frac{n}{t} 1_{(t)}^T \quad (2.6)$$

Then (2.2) and (2.6) yield

$$X_i^T X_i = \frac{n}{t} I_{(t)} \quad . \tag{2.7}$$

Therefore, the normal equations (2.3), (2.4) and (2.5) simplify to

$$\sum_1 X_i^T y_i = (\frac{n}{t} \sum_1 \mu_i) 1_{(t)} + \frac{pn}{t} \hat{\tau} + \sum_2 X_i^T X_{i-1} \hat{\rho} + \sum_1 X_i^T \hat{\beta} \quad , \tag{2.8}$$

$$\sum_2 X_{i-1}^T y_i = (\frac{n}{t} \sum_2 \mu_i) 1_{(t)} + \sum_2 X_{i-1}^T X_i \hat{\tau} + \frac{(p-1)n}{t} \hat{\rho} + \sum_2 X_{i-1}^T \hat{\beta} \quad , \tag{2.9}$$

$$\sum_1 y_i = (\sum_1 \mu_i) 1_{(n)} + \sum_1 X_i \hat{\tau} + \sum_2 X_{i-1} \hat{\rho} + p \hat{\beta} \quad . \tag{2.10}$$

We shall see in the later sections of this chapter that the terms orthogonality, balance, efficiency factors and relative efficiency can be defined in terms of the matrices $\sum_1 X_i$, $\sum_2 X_{i-1}$, $\sum_2 X_{i-1}^T X_{i-1}$, $\sum_1 X_i^T X_i$, $\sum_2 X_{i-1}^T X_i$. This is the advantage of the model considered here.

2.5 Orthogonality.

In this section we examine the question of orthogonality for designs satisfying equations (2.6) and (2.7). It is sufficient to consider ~~to the restrictions on the model we can only estimate~~

$$T_{(t)} \tau = (I_{(t)} - {}^{J_{(t)}}/_{t}) \tau \quad ,$$

$$T_{(t)} \rho = (I_{(t)} - {}^{J_{(t)}}/_{t}) \rho \quad ,$$

$$T_{(n)} \beta = (I_{(n)} - {}^{J_{(n)}}/_{n}) \beta \quad ,$$

where $T_{(m)} = I_{(m)} - {}^{J_{(m)}}/_{m}$, a symmetric idempotent matrix, and $J_{(t)}$ are defined in Appendix A at the end of the thesis. Therefore the information matrix on direct effects, residual effects and subjects is

$$\begin{bmatrix} T_{(t)} \sum_1 X_i^T X_{i.} T_{(t)} & T_{(t)} \sum_2 X_i^T X_{i-1.} T_{(t)} & T_{(t)} \sum_1 X_{i.}^T T_{(n)} \\ & T_{(t)} \sum_2 X_{i-1}^T X_{i-1.} T_{(t)} & T_{(t)} \sum_2 X_{i-1.}^T T_{(n)} \\ \text{symmetric} & & p T_{(n)} \end{bmatrix}$$

The information matrix of direct effects and subjects, both adjusted for residual effects, is

$$\begin{bmatrix} T_{(t)} A T_{(t)} & T_{(t)} B T_{(n)} \\ T_{(n)} B^T T_{(t)} & T_{(n)} C T_{(n)} \end{bmatrix}$$

where

$$A = \sum_1 X_i^T X_i - \sum_2 X_i^T X_{i-1} (\sum_2 X_{i-1}^T X_{i-1})^{-1} \sum_2 X_{i-1}^T X_i ,$$

$$B = \sum_1 X_i^T - \sum_2 X_i^T X_{i-1} (\sum_2 X_{i-1}^T X_{i-1})^{-1} \sum_2 X_{i-1}^T ,$$

$$C = p I_{(n)} - \sum_2 X_{i-1} (\sum_2 X_{i-1}^T X_{i-1})^{-1} \sum_2 X_{i-1}^T .$$

Therefore, we have the following definition.

Def$^{\underline{n}}$ 2.1 : Direct effects adjusted for residual effects are orthogonal to subjects if

$$T_{(t)} \left\{ \sum_1 X_i^T - \sum_2 X_i^T X_{i-1.} (\sum_2 X_{i-1}^T X_{i-1})^{-1} \sum_2 X_{i-1}^T \right\} T_{(n)} = 0 .$$

The condition for this is that the term in braces = k $J_{(t\times n)}$ for some constant k . Designs constructed by Berenblut (1964) satisfy this condition. We shall see later that other designs also satisfy this condition. The designs in section 2.2 and the new A-class designs in Section 2.3 do not have this property.

The information matrix of residual effects and subjects, both adjusted for direct effects, is

$$\begin{bmatrix} T_{(t)} \, D \, T_{(t)} & T_{(t)} \, E \, T_{(n)} \\ T_{(n)} \, E^T T_{(t)} & T_{(n)} \, F \, T_{(n)} \end{bmatrix} ,$$

where

$$D = \sum_2 X_{i-1}^T X_{i-1} - \sum_2 X_{i-1}^T X_{i.} (\sum_1 X_i^T X_i)^{-1} \sum_2 X_i^T X_{i-1} ,$$

$$E = \sum_2 X_{i-1}^T - \sum_2 X_{i-1}^T X_{i.} (\sum_1 X_i^T X_i)^{-1} \sum_1 X_i ,$$

$$F = p \, I_{(n)} - \sum_1 X_{i.} (\sum_1 X_i^T X_i)^{-1} \sum_1 X_i^T .$$

Therefore we have the second definition.

Def$^{\underline{n}}$ 2.2 : Residual effects adjusted for direct effects are orthogonal to subjects if

$$T_{(t)} \left\{ \sum_2 X_{i-1}^T - \sum_2 X_{i-1}^T X_{i.} (\sum_1 X_i^T X_i)^{-1} \sum_1 X_i \right\} T_{(n)} = 0 .$$

Again the condition is that the term in braces $= k \, J_{(t \times n)}$ for some constant k . This condition is satisfied in designs described by Lucas (1957) and in the A-class designs in section 2.3.

Finally, the information matrix of direct effects and residual effects, both adjusted for subjects is,

$$\begin{bmatrix} T_{(t)} \, G \, T_{(t)} & T_{(t)} \, H \, T_{(n)} \\ T_{(n)} \, H^T T_{(t)} & T_{(n)} \, K \, T_{(n)} \end{bmatrix} ,$$

where

$$G = \sum_1 X_i^T X_i - \frac{1}{p} \sum_1 X_i^T \sum_1 X_i \quad ,$$

$$H = \sum_2 X_i^T X_{i-1} - \frac{1}{p} \sum_1 X_i^T \sum_2 X_{i-1} \quad ,$$

$$K = \sum_2 X_{i-1}^T X_{i-1} - \frac{1}{p} \sum_2 X_{i-1}^T \sum_2 X_{i-1} \quad .$$

Therefore, we have the third definition.

Def$^{\underline{n}}$ 2.3 : Direct effects adjusted for subjects are orthogonal to residual effects if

$$T_{(t)} \left\{ \sum_2 X_i^T X_{i-1} - \frac{1}{p} \sum_1 X_i^T \sum_2 X_{i-1} \right\} T_{(t)} = 0 \quad .$$

The condition is that the term in braces = $k\, J_{(t)}$ for some constant k. Designs by Berenblut (1964), Lucas (1957), Patterson and Lucas (1959) and the new A-class designs have this property. Note that, in general, omission of one or more periods of these designs will result in a loss of the orthogonality property. We shall see in later chapters that other designs, including those constructed by Patterson (1970), also satisfy the condition. The present thesis deals with a wide class of designs, including those constructed by Berenblut (1964) and Patterson (1970), all with orthogonality of direct and residual effects.

2.6 Reparameterization of model.

In some situations, interest may be centred on the estimation of a set of orthogonal normalised direct effect contrasts, $\underset{(t-1)\times t}{T} \tau$ instead of τ where

$$T\,T^T = I_{(t-1)}$$

$$T^T T = I_{(t)} - {}^{J_{(t)}}/_{t} \text{ , an idempotent matrix.}$$

Similarly, we may be interested in estimating $\underset{(x\times t)}{U} \rho$, $x \le t-1$

where

$$U U^T = I_{(x)}$$

$$U^T U = I_{(t)} - {}^{J_{(t)}}/_{t} \quad \text{for} \quad x = t-1$$

$$U^T U = \text{some idempotent matrix for} \quad x < t-1 \quad .$$

The case with $x < t-1$ is considered because we may not be interested in estimating the full set of residual effect contrasts. (See Berenblut (1967b, 1968)). For example, consider a particular case of (2.1) in which we are interested in estimating $T_{QU}\ \tau$ and $U_L\ \rho$, where T_{QU} is the $(t-1)\times t$ matrix of normalised contrasts of linear direct effect, quadratic direct effect, ..., etc. U_L is the $(1\times t)$ row vector of normalised contrast of linear residual effect. For $t=4$, we have

$$T_{QU} = \begin{bmatrix} {}^{-3}/\sqrt{20} & {}^{-1}/\sqrt{20} & {}^{1}/\sqrt{20} & {}^{3}/\sqrt{20} \\ {}^{1}/_{2} & {}^{-1}/_{2} & {}^{-1}/_{2} & {}^{1}/_{2} \\ {}^{-1}/\sqrt{20} & {}^{3}/\sqrt{20} & {}^{-3}/\sqrt{20} & {}^{1}/\sqrt{20} \end{bmatrix}, \quad U_L = \left({}^{-3}/\sqrt{20} \quad {}^{-1}/\sqrt{20} \quad {}^{1}/\sqrt{20} \quad {}^{3}/\sqrt{20} \right)$$

Then for designs by Berenblut (1967b, 1968), it can be shown that direct effects, adjusted for residual effects, are orthogonal to subjects, and direct effects, adjusted for subjects, are orthogonal to residual effects.

2.7 Balance.

In the past, balance has been defined as the design property ensuring that differences between the elements of $\hat{\tau}$ have equal variances and differences between the elements of $\hat{\rho}$ also have equal variances. For example, see Patterson and Lucas (1959). General conditions for balance are also given by Patterson and Lucas (1959).

A formal definition of balance will be given here.

Def$^{\underline{n}}$ 2.4 : A design is said to be balanced with respect to a set of normalised treatment contrasts if the variances of these contrasts, adjusted for all other effects in the model considered, are equal.

In particular, we have here

Def$^{\underline{n}}$ 2.5 : A design with t treatments is balanced for a specified set of $(t-1)$ normalised direct effect contrasts, $T_3 \tau$, if the variances of these contrasts, adjusted for residual effect contrasts, $\underset{(t-1)\times t}{U} \rho$ (see Section 2.6), and subjects, are equal.

This can also be expressed in terms of the variance-covariance matrix of $T_3 \tau$. This matrix is given by S_k, where

$$S_k = \left\{ T_3 \left[G - HU^T (UKU^T)^{-1} UH^T \right] T_3^T \right\}^{-1} .$$

Matrices G, H and K are given in Section 2.5.

The design is balanced if S_k is symmetric with all diagonal elements equal to k. The variances would then be equal to $k\sigma^2$. Designs by Cochran et al (1941), Williams (1949), Lucas (1957), Patterson (1951, 1952), Patterson and Lucas (1959), Berenblut (1964) and new A-class designs are balanced for any set of normalised direct effect contrasts, $T_3 \tau$, since the term in square brackets of the above expression,

$[\quad] = k_1 I_{(t)} + k_2 J_{(t)}$ for constants k_1, k_2 depending on the design, so that $T_3[\quad] T_3^T$ is a symmetric matrix. If the contrasts are orthogonal, that is, $T_3 \tau = T \tau$ (see Section 2.6), then $T_3 \hat{\tau}$ are uncorrelated in the above designs since $T[\quad] T^T = k_1 I_{(t-1)}$. Note that the designs by Berenblut (1967b, 1968) and Latin square design in Table 2.1 are not balanced for any set of contrasts, $T_3 \tau$. For the model where we are interested in estimating $T_{QU} \tau$ and $U_L \rho$ (see Section 2.6), then the designs by Berenblut (1967b, 1968) are balanced for contrasts $T_{QU}\tau$. Designs by Davis and Hall (1969) are not, in general, balanced for every set of contrasts, $T_3\tau$.

Similarly, we have the definition.

Def$^{\underline{n}}$ 2.6 : A design with t treatments is balanced for a set of $(t-1)$ normalised residual effect contrasts, $U_3\rho$, if the variance-covariance matrix of $U_3\rho$, given by S_{k_3}, is such that

$S_{k_3}=\left\{U_3\,[K - H^T T^T (T G T^T)^{-1}\, TH]\, U_3^T\right\}^{-1}$, a symmetric matrix with all diagonal elements equal to k_3 which depends on the design. Matrices G, H and K are given in section 2.5 and matrix T in section 2.6 . The variances are then equal to $k_3\sigma^2$. Since the designs by Cochran et al (1941), Williams (1949), Lucas (1957), Berenblut (1964) and new A-class designs have

$$G = k_4\, I_{(t)} + k_5\, J_{(t)} ,$$

$$H = k_6\, I_{(t)} + k_7\, J_{(t)} ,$$

$$K = k_8\, I_{(t)} + k_9\, J_{(t)} ,$$

for constants k_4, k_5, k_6, k_7, k_8, and k_9 depending on the design, then it is obvious that these designs are also balanced for any set of normalised residual effect contrasts, $U_3\rho$. Similarly, it can be shown that designs by Patterson (1951, 1952) and Patterson and Lucas (1959) also have this property. If the contrasts are orthogonal, that is, $U_3\rho = U\rho$ (see section 2.6), then $U_3\hat{\rho}$ are uncorrelated in the above designs. Again, the designs by Berenblut (1967b, 1968) and the Latin square design in Table 2.1 are not balanced for every set of contrasts, $U_3\rho$. Of course, for the particular model considered in section 2.6, it is trivial to show that the designs by Berenblut (1967b, 1968) are balanced for the single contrast, $U_L\rho$. Designs by Davis and Hall (1969) are not, in general, balanced for every set of contrasts, $U_3\rho$.

2.8 Efficiency factor of a design.

We now consider the efficiency factor of a design for the estimation of a direct effect contrast or a residual effect contrast. They are defined below.

Def$^{\underline{n}}$ 2.7 : The efficiency factor of design X for the estimation of a normalised direct effect contrast is defined as the ratio of its variance in a Latin square design, L_1, with its treatments regarded as direct effects, with no residual effects, to its variance in design X.

In particular, if the normalised direct effect contrast is $\underset{(1\times t)}{T_i}\,\tau$ in a set of (t-1) orthogonal normalised direct effect contrasts, $T\tau$, where $T = \begin{bmatrix} T_1 \\ \vdots \\ T_i \\ \vdots \\ T_{(t-1)} \end{bmatrix}$, then the efficiency factor for the estimation of $T_i\tau$ is

$$\frac{\text{cell } (i,i) \text{ of } (T \sum_1 X_{i.}^T X_{i.} T^T)^{-1} \ \sigma_{L_1}^2}{\text{cell } (i,i) \text{ of } \left\{ T\,[G - HU^T(UKU^T)^{-1}UH^T\,]\,T^T \right\}^{-1} \sigma_X^2},$$

where matrices G, H and K are given in Section 2.5, and σ_X^2 and $\sigma_{L_1}^2$ are the error variances of design X and the Latin square design, L_1, respectively.

The above expression is general for any design. However, for designs in which the estimated contrasts, $T\hat{\tau}$, are uncorrelated, the above expression simplifies to

$$\frac{(T_i \sum_1 X_{i.}^T X_{i.} T_i^T)^{-1} \ \sigma_{L_1}^2}{\left\{ T_i\,[G - HU^T(UKU^T)^{-1}UH^T\,]\,T_i^T \right\}^{-1} \sigma_X^2},$$

that is, the ratio of the variance of $T_i\hat{\tau}$ in the Latin square design, L_1, to the variance of $T_i \hat{\tau}$ in design X. The ratio is constant for any $T_i \hat{\tau}$ if design X is balanced for the set of contrasts, $T \hat{\tau}$.

Similarly, we have

Def$^{\underline{n}}$ 2.8 : The efficiency factor of design X for the estimation of a normalised residual effect contrast is defined as the ratio of its variance in a Latin square design, L_2, with its treatments regarded as residual effects, and no direct effects, to its variance in design X.

In particular, if the normalised residual effect contrast is $\underset{(1\times t)}{U_i}\,\rho$ in a set of $(t-1)$ orthogonal normalised residual effect contrasts, $U\rho$, where

$$U = \begin{bmatrix} U_1 \\ \vdots \\ U_i \\ \vdots \\ U_{(t-1)} \end{bmatrix},$$

then the efficiency factor for the estimation of $U_i \rho$ is

$$\frac{\text{cell } (i, i) \text{ of } (U \sum_2 X_{i-1}^T \; X_{i-1.} U^T)^{-1} \; \sigma_{L_2}^2}{\text{cell } (i, i) \text{ of } \left\{ U \, [K - H^T T^T (TGT^T)^{-1} TH] \, U^T \right\}^{-1} \sigma_X^2}$$

where matrices G, H and K are given in Section 2.5 and $\sigma_{L_2}^2$ is the error variance of the Latin square design, L_2. If the estimated contrasts, $U\hat{\rho}$, are uncorrelated, the above expression simplifies to

$$\frac{(U_i \sum_2 X_{i-1}^T \; X_{i-1.} U^T)^{-1} \; \sigma_{L_2}^2}{\left\{ U_i \, [K - H^T T^T (TGT^T)^{-1} TH] \, U_i^T \right\}^{-1} \sigma_X^2},$$

which has the same value for any contrast, $U_i \rho$, if design X is balanced for the set of contrasts, $U\rho$.

Generally, we assume $\sigma_{L_1}^2 = \sigma_{L_2}^2 = \sigma_X^2$ in all the above expressions. The efficiency factors for the estimation of $T_i \tau$ and $U_i \rho$ of some designs which are balanced for any set of contrasts, $T\tau$ or $U\rho$ and where the estimated contrasts, $T\tau$, are uncorrelated and the estimated contrasts, $U\hat{\rho}$, are also uncorrelated, are given in Table 2.11.

Table 2.11 Efficiencies of any contrast of direct and residual effects of some designs.

Design	Number of Subjects	Number of Periods	Efficiency Direct	Efficiency Residual
1. Cochran et al (1941)	$t(t-1)$	t	$1-\frac{1}{t^2-t-1}$	$1-\frac{2}{t(t-1)}$
2. Williams (1949)				
t odd	$2t$	t	$1-\frac{1}{t^2-t-1}$	$1-\frac{2}{t(t-1)}$
t even	t	t	$1-\frac{1}{t^2-t-1}$	$1-\frac{2}{t(t-1)}$
3. Lucas (1957)			$1-\frac{1}{(t+1)^2}$	1
4. Patterson (1951, 1952) $t=7$	14	4	0.798	0.731
5. Patterson and Lucas (1959), $t=7$	14	5	0.84	0.90
6. Federer and Atkinson (1964)	$t(t-1)$	$qt+1$	$1-\frac{qt}{(t-1)^2\ (qt+1)}-\frac{1}{(2t+1)^2}$	$1-\frac{qt+1}{(t-1)^2\ (qt+2)}$
7. Berenblut (1964)	t^2	$2t$	1	$1-\frac{1}{2t(2t-1)}$
8. Berenblut (1967, 1968)* $t=4, 5$	$2t$	t	1	$1-\frac{1}{t(t-1)}$
9. A-class designs	$t(t-1)$	$qt+1$	$1-\frac{1}{(qt+1)^2}$	1

* We assume the particular model of Section 2.6 for this design.

2.9 Relative efficiency.

In this section, we consider the comparison of any two designs in the estimation of a treatment contrast. Such comparison is made by calculating the relative efficiency of one design compared to the other, defined below.

Def$^{\underline{n}}$ 2.9 : The relative efficiency of design X compared to design Y in the estimation of a normalised treatment (direct or residual effect) contrast is the ratio of the product of the number of observations and the variance of that contrast in design Y to the corresponding quantity in design X , that is,

$$E_{XY} = \frac{\sigma_Y^2/E_Y r_Y \cdot p_Y n_Y}{\sigma_X^2/E_X r_X \cdot p_X n_X}$$

where E_Y is the efficiency in the estimation of the normalised treatment contrast in design Y ,

p_Y is the number of periods in design Y ,

n_Y is the number of subjects in design Y ,

r_Y is the number of replications of each treatment in design Y ,

σ_Y^2 is the error variance in design Y .

The corresponding quantities E_X, p_X, n_X, r_X and σ_X^2 for design X are similarly defined, Generally, we assume $\sigma_X^2 = \sigma_Y^2$.

2.9.1 Relative efficiency of A-class designs compared to designs by Federer and Atkinson (1964).

In this subsection, we compare A-class designs and the designs by Federer and Atkinson (1964). The relative efficiency of A-class designs and those of Federer and Atkinson (1964) in the estimation of a normalised direct or residual effect contrast is given in Table 2.12 .

Table 2.12 Relative efficiency of A-class designs by Federer and Atkinson (1964).

Number of periods	Design X	Design Y	Relative efficiency Direct effects	Residual effects
qt+1	A-class	Federer and Atkinson	$\dfrac{1}{1-\dfrac{qt+1}{(t-1)^2\,(qt+2)}}$	$\dfrac{1}{1-\dfrac{qt+1}{(t-1)^2\,(qt+2)}}$
4	"	"	$5/4$	$5/4$
7	"	"	$32/25$	$32/25$

Thus A-class designs are more efficient in the estimation of a normalised direct or residual effect contrast than the designs by Federer and Atkinson (1964) for any number of treatments, t , and integral values of s and q .

2.9.2 Relative efficiency of A-class designs compared to the designs by Berenblut (1964)

We now compare A-class designs and the designs by Berenblut (1964). The relative efficiency of A-class designs compared to the designs by Berenblut (1964) in the estimation of a normalised direct or residual effect contrast is given in Table 2.13.

Table 2.13 Relative efficiency of A-class designs with the designs by Berenblut (1964).

Number of Treatments	Number of Periods in design X	Design X	Design Y	Relative efficiency	
				Direct effects	Residual effects
t	$qt+1$	A-class design	Berenblut (1964)	$1-\frac{1}{(qt+1)^2}$	$\frac{2t^2q}{(2t-1-\frac{1}{2t})(qt+1)}$
3	4	"	"	$^{15}/_{16}$	$^{27}/_{29}$
3	7	"	"	$^{48}/_{49}$	$^{216}/_{203}$
4	5	"	"	$^{24}/_{25}$	$^{256}/_{275}$
4	9	"	"	$^{80}/_{81}$	$^{512}/_{495}$

Thus the designs by Berenblut (1964) are more efficient in the estimation of a normalised direct effect contrast than A-class designs with any number of periods, $qt+1$. They are also more efficient in the estimation of a normalised residual effect contrast than A-class designs with $(t+1)$ periods. However, A-class designs are more efficient when their number of periods is $qt+1$, $q \geq 2$.

Chapter Three

Designs for model with non-additive direct effects and residual effects.

3.1 Introduction

In this chapter we consider a model with provision for not only direct and residual effects but also direct × first residual interaction. This model is used for designs with t treatments involving t^2 subjects and $2t$ periods. Some properties of this class of designs are shown.

3.2 Preliminaries

3.2.1 Order of combinations of treatment levels in periods i, i'

The following standard order of combinations of treatment levels in periods i, i' will be used.

Period								
i	1	1	1 ...	2	2	2 ...	l	... t
i'	1	2	3 ...	1	2	3 ...	l'	... t
Combination	1	2	3 ...	$t+1$	$t+2$	$t+3$ ...	$t(l-1)+l'$	... t^2

3.2.2 Forward and Reverse order

We are specially interested in the cases in which $i' = i+1$ or $i-1$. In the former case we refer to forward order; in the latter to reverse order.

Forward order is used mainly in description of designs. For example, we may number the subjects so that their treatments in periods 1 and 2 are in forward order:

Period	Subject 1	2	3	4	...	t^2
1	1	1	1	1	...	t
2	1	2	3	4	...	t

Reverse order is used in referring to tables of effects. For example, suppose we have a table of means with entry (l, m) giving the mean for subjects receiving treatment level l in the current period following treatment level m in the preceding period. The entries are numbered 1 to t^2 as follows:

				m		
		1	2	3		t
	1	1	2	3	...	t
	2	t+1	t+2	t+3	...	2t
l	3	⋮				⋮
	t	t^2-t+1	.	.	...	t^2

Then direct effects are given by comparisons between row means. Residual effects are given by comparisons between column means. Interactions are given by row × column comparisons.

3.2.3 Second-order incidence matrix

We now define a second-order incidence matrix. This should be distinguished from the first-order incidence matrix defined in Chapter 2. The second-order incidence matrix $X_{i,i'}$ is an $n \times t^2$ matrix with cell $(u, v) = 1$ if combination $v(= t(l-1) + l')$ is on subject u in periods i, i' where l, l' are the levels in periods i, i', respectively, and cell $(u, v) = 0$ otherwise.

Second-order incidence matrices are available for any i, i' such that $i \neq i'$ but we are particularly interested in $X_{i,i-1}$ for $i = 2, \ldots, p$. For example, the matrices X_{21} and X_{35} for the design by Berenblut (1964) in Table 2.8 are given by

Combination

$$X_{21} = \begin{array}{c|c} & \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \begin{array}{r} 1 \\ 2 \\ 3 \\ 4 \\ \text{subject } 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Combination

$$X_{35} = \begin{array}{c|c} & \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \begin{array}{r} 1 \\ 2 \\ 3 \\ 4 \\ \text{subject } 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Note that $X_{i,i+1} \neq X_{i+1,i}$ but that either can be obtained from the other by a permutation of columns. Thus $X_{i,i+1}\,\Pi = X_{i+1,i}$ (3.1) where Π is a permutation matrix in which cell $(f, g) = 1$. if f, g are such that

$$f = (a-1)t + b$$
$$g = (b-1)t + a$$

and 0 otherwise (a, b are positive integers). For example when $t = 3$ Π is given by

$$\left[\begin{array}{ccc|ccc|ccc}
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\hline
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\hline
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{array}\right]$$

3.2.4 Relationship between second- and first-order incidence matrices

In this subsection, we examine the relationship between second- and first-order incidence matrices. The definition of (right) direct product of matrices and some theorems on direct product are given in Appendix B at the end of the thesis.

We define

$$\underset{(t^2 \times t)}{Q_1} = I_{(t)} \otimes 1_{(t)} \tag{3.2}$$

$$\text{and} \quad Q_2 = 1_{(t)} \otimes I_{(t)} ,$$

$$\text{with } \Pi Q_1 = Q_2 , \quad \Pi \text{ defined in subsection 3.2.3} .$$

Then the (first-order) incidence matrix X_i is given by

$$X_i = X_{i+1,i}\, Q_2 \quad (= X_{i,i+1}\, Q_1) \tag{3.3}$$

or by

$$X_i = X_{i,i-1}\, Q_1 \quad (= X_{i-1,i}\, Q_2) .$$

For example, the matrix X_2 for the design of Table 2.8 is given by

$$X_2 = \begin{array}{c c} & \text{treatment level} \\ & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ \text{observation}\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

Then it can be verified that $X_2 = X_{21}\, Q_1$ where X_{21} for the same design is given in subsection 3.2.3 .

The coincidence matrix $X_{i-1}^T\, X_i$ (defined in Chapter Two) is, therefore, given by

$$X_{i-1}^T\, X_i = Q_2^T\, X_{i,i-1}^T\, X_{i,i-1}\, Q_1 \quad . \tag{3.4}$$

3.2.5 Associated incomplete block (a.i.b.) design

A class of incomplete block designs associated with changeover designs is considered in this subsection. The associated incomplete block design is obtained by replacing treatment levels l in period i and l' in period $i-1$ of the same subject in a changeover design by $v\,(= t(l-1) + l')$, $(i = 2, 3, \ldots, p)$. Subjects are regarded as 'blocks'. For example, the associated incomplete block design of the design by Berenblut (1964) in Table 2.8 is given in Table 3.1 .

Table 3.1. Associated incomplete block design of the design in Table 2.8.

		Subject								
		I	II	III	IV	V	VI	VII	VIII	IX
Period	1	1	5	9	4	8	3	7	2	6
	2	4	8	3	5	9	1	6	7	2
	3	5	9	1	8	3	4	2	6	7
	4	8	3	4	9	1	5	7	2	6
	5	9	1	5	3	4	8	6	7	2

The incidence matrix of the a.i.b. design is given by

$$N = \sum_{i=2}^{p} X_{i,i-1}^{T} \qquad (3.5)$$

where cell (v, u) gives the frequency with which combination v occurs in subject u. The concurrence matrix is NN^T with cell (v, w) giving the frequency with which combinations v, w occur together in a subject. For the a.i.b. design in Table 3.1 above, its incidence matrix, N, is given in Table 3.2 and its concurrence matrix, NN^T, in Table 3.3.

Table 3.2. Incidence matrix, N, of the a.i.b. design in Table 3.1.

Subject

$$N = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} \begin{array}{c} \begin{array}{ccccccccc} \mathrm{I} & \mathrm{II} & \mathrm{III} & \mathrm{IV} & \mathrm{V} & \mathrm{VI} & \mathrm{VII} & \mathrm{VIII} & \mathrm{IX} \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 2 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

combination

Table 3.3 . Concurrence matrix, NN^T, of the a.i.b. design in Table 3.1 .

$$NN^T = \text{combination}\begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7\\8\\9\end{array}\begin{array}{c}\text{Combination}\\ \begin{array}{ccccccccc}1&2&3&4&5&6&7&8&9\end{array}\\ \left[\begin{array}{ccccccccc}5&0&4&4&4&0&0&4&4\\0&9&0&0&0&8&8&0&0\\4&0&5&4&4&0&0&4&4\\4&0&4&5&4&0&0&4&4\\4&0&4&4&5&0&0&4&4\\0&8&0&0&0&9&8&0&0\\0&8&0&0&0&8&9&0&0\\4&0&4&4&4&0&0&5&4\\4&0&4&4&4&0&0&4&5\end{array}\right]\end{array}$$

3.2.6 Some algebraic results.

Here we obtain some algebraic results which are useful for the rest of the chapter. Let A, B be general matrices. Then the following hold:

1. $1^T \otimes B = (B\ B\ B\ \ldots)$

$$\therefore A(1^T \otimes B) = (AB\ AB\ AB\ \ldots) = 1^T \otimes AB . \tag{3.6}$$

2. $$\Pi(1^T \otimes \underset{(t^2 \times t)}{B}) = 1^T \otimes \Pi B \tag{3.7}$$

but $$(1^T \otimes \underset{(t^2 \times t)}{B})\,\Pi = B \otimes 1^T \tag{3.8}$$

and $$\Pi(\underset{(t \times t)}{A} \otimes \underset{(t \times t)}{B})\,\Pi = B \otimes A . \tag{3.9}$$

In Appendix 3.1 at the end of the chapter, we define

$$R_1 = \frac{1}{t} Q_1 \otimes 1^T_{(t)} = I_{(t)} \otimes \frac{1}{t} J_{(t)}$$

$$\text{and } R_2 = \frac{1}{t} 1^T_{(t)} \otimes Q_2 = \frac{1}{t} J_{(t)} \otimes I_{(t)}$$

Then the following results hold:

(1a) $\quad \Pi R_1 \Pi = R_2 .$ (3.10)

Proof: $R_1 = I_{(t)} \otimes \frac{1}{t} J_{(t)}$

$\therefore \quad \Pi R_1 \Pi = \frac{1}{t} J_{(t)} \otimes I_{(t)} = R_2$ as $I_{(t)}, J_{(t)}$ are of size txt.

Alternative proof: $R_1 = \frac{1}{t} Q_1 \otimes 1^T_{(t)}$.

$$\Pi R_1 = \frac{1}{t} (\Pi Q_1) \otimes 1^T_{(t)}$$

$$= \frac{1}{t} Q_2 \otimes 1^T_{(t)} .$$

$$\Pi R_1 \Pi = (\frac{1}{t} Q_2 \otimes 1^T_{(t)}) \Pi$$

$$= 1^T_{(t)} \otimes \frac{1}{t} Q_2 = R_2 .$$

(1b.) $\quad \Pi R_2 \Pi = R_1 .$ (3.11)

Proof: $R_2 = \Pi R_1 \Pi$.

$\therefore \quad \Pi R_2 \Pi = \Pi\Pi R_1 \Pi\Pi$

$$= R_1 \text{ since } \Pi\Pi = I_{(t^2)} .$$

(2) $\quad 1^T_{(t)} \otimes X_i = tX_{i+1,i} R_2 .$ (3.12)

Proof: $X_i = X_{i+1,i} Q_2$.

$\therefore \ 1^T_{(t)} \otimes X_i = 1^T_{(t)} \otimes (X_{i+1,i} Q_2)$

$$= X_{i+1,i}(1^T_{(t)} \otimes Q_2)$$

$$= t X_{i+1,i} R_2 .$$

(3) $\quad X_i \otimes 1^T_{(t)} = t X_{i,i-1} R_1 .$ (3.13)

Proof: $X_i = X_{i,i-1} Q_1$.

$\therefore \ X_i \otimes 1^T_{(t)} = (X_{i,i-1} Q_1) \otimes 1^T_{(t)}$

$$= X_{i,i-1}(Q_1 \otimes 1^T_{(t)})$$

$$= t X_{i,i-1} R_1 .$$

3.3 Model with provision for direct × first residual interaction

The model can be expressed as

$$Ey_1 = \mu_1 1_{(n)} + \frac{1}{t}(X_1 \otimes 1^T_{(t)})\alpha + \beta ,$$

$$Ey_i = \mu_i 1_{(n)} + X_{i,i-1}\alpha + \beta \qquad i = 2, 3, \ldots, p \qquad (3.14)$$

$$\text{or} \quad Ey_i = \mu_i 1_{(n)} + X_{i-1,i}\Pi\alpha + \beta ,$$

where the subjects are arranged in some standard order,

β is a vector of subject effects in this order

y_i is a vector of yields also in this order,

elements of y_i and β both follow this order for all i ,

α is the $(t^2 \times 1)$ vector of treatment combination means in reverse order (see subsection 3.2.2),

$$X_{10} = \frac{1}{t}(X_1 \otimes 1^T_{(t)}) \qquad (3.15)$$

$$= \frac{1}{t}(1^T_{(t)} \otimes X_1)\Pi = X_{01}\Pi \qquad (3.16) \text{ from } (3.7)$$

A separate equation is given for period 1 since no estimates of residual effects or direct × residual interaction are available for this period. We assume that α and β are fixed effects in the model. We also assume the same error model here as for the model in Chapter 2 for direct and residual effects only, that is,

$$\text{Var}(y) = (I_{(p)} \otimes I_{(n)})\sigma^2$$

where $y = \begin{pmatrix} y_1 \\ \vdots \\ y_i \\ \vdots \\ y_p \end{pmatrix}$.

3.3.1 Normal equations.

The normal equations for estimating α and β are

$$\sum_1 X^T_{i,i-1} y_i = \sum_1 X^T_{i,i-1} 1_{(n)} \mu_i + \left\{ X^T_{10} X_{10} + \sum_2 X^T_{i,i-1} X_{i,i-1} \right\} \hat{\alpha} + (N_0 + N)\hat{\beta} ,$$

(3.19)

$$\sum_1 y_i = 1_{(n)} \sum_1 \mu_i + (N_0^T + N^T)\,\hat{\alpha} + p\hat{\beta} \;,$$

where $N_0 = X_{10}^T$; $N = \sum_2 X_{i,i-1}^T$. (3.20)

Note that $X_{10}^T\, X_{10} = \frac{1}{t}\,(X_1^T \otimes 1_{(t)})\; \frac{1}{t}\,(X_1 \otimes 1_{(t)}^T)$

$$= \frac{1}{t}\,(X_1^T\, X_1) \otimes \frac{1}{t}\, J_{(t)} \;. \qquad (3.21)$$

3.3.2 Information.

1. The information matrix, assuming subject effects are known to be zero, is

$$C_0 = X_{10}^T\, X_{10} + \sum_2 X_{i,i-1}^T\, X_{i,i-1} \;. \qquad (3.22)$$

Therefore, the information matrix on direct effects is $T_1 C_0 T_1$.
The information matrix on residual effects is $T_2 C_0 T_2$.
The information matrix on interaction is $T_{12} C_0 T_{12}$.
(The matrices T_1, T_2 and T_{12} are defined in Appendix 3.1 at the end of the chapter. The matrices T_0, R_0, R_1, R_2 and R_{12} are also defined in Appendix 3.1 . These matrices will be used in the later sections of this chapter).

It is desirable to make $T_1 C_0 T_2$, $T_1 C_0 T_{12}$, $T_2 C_0 T_{12}$ all zero. Estimates are then orthogonal although not necessarily statistically independent. When subject effects are known to be zero, the orthogonality conditions $T_1 C_0 T_2 = 0$, etc., ensure statistical independence. But if subject effects are not zero, direct effects, residual effects and interaction are not necessarily independent even when the orthogonality conditions are satisfied.

2. The information matrix eliminating subjects is

$$C = X_{10}^T\, X_{10} + \sum_2 X_{i,i-1}^T X_{i,i-1} - \frac{1}{p}(N_0 + N)(N_0^T + N^T) \;. \qquad (3.23)$$

3.4 SF designs with t treatments, t^2 subjects and $2t$ periods.

In this and the remaining sections of this chapter, we consider the class of designs with each combination of treatment levels in periods $i-1$, i on exactly one subject, where $i = 2, 3, \ldots, 2t$ and with each treatment level twice on each subject. This is not the only class of designs with serial factorial property. There are other classes of designs with such property; for example, the class of designs with each combination of treatment levels in periods $i-2$, $i-1$, i on exactly one subject, where $i = 3, 4, \ldots, 3t$ and with each treatment level thrice on each subject. These classes of designs would not be considered further in this thesis. Therefore for convenience, throughout this thesis, we shall refer to the above class of designs with t treatments, t^2 subjects and $2t$ periods as serial factorial (SF in short) designs. These SF designs include, as special cases, designs constructed by Berenblut (1964) and Patterson (1970). The construction of a wider class of SF designs will be considered in Chapter 4.

3.4.1 Combinatorial properties of SF designs.

The combinatorial properties of SF designs are

(I). Each combination of levels in periods $i-1$, i occurs on one subject $(i = 2, 3, \ldots, 2t)$. Therefore, the $X_{i,i-1}$ $(i = 2, 3, \ldots, 2t)$ are permutation matrices and hence $X^T_{i,i-1} X_{i,i-1} = I_{(t^2)}$, $i = 2, 3, \ldots, 2t$. The following algebraic relationships hold: (3.24)

$$1. \quad 1^T_{(t^2)} X_i = t1^T_{(t)} \quad ; \quad X_i 1_{(t)} = 1_{(t^2)} \tag{3.25}$$

$$X^T_i X_i = tI_{(t)} \quad ; \quad J_{(t)} X^T_i = J_{(t \times t^2)} \quad .$$

2. $$X^T_{i-1} X_i = J_{(t)} . \qquad (3.26) \text{ from } (3.4)$$

3. $$J_{(t^2)}(1^T_{(t)} \otimes X_i) = tJ_{(t^2)} ; (1_{(t)} \otimes X^T_i) J_{(t^2)} = tJ_{(t^2)} \qquad (3.27)$$

4. $$1^T_{(t^2)} X_{i,i-1} = 1^T_{(t^2)} ; \quad X_{i,i-1} 1_{(t^2)} = 1_{(t^2)}$$

$$J_{(t^2)} X_{i,i-1} = J_{(t^2)} ; \quad X_{i,i-1} J_{(t^2)} = J_{(t^2)} \qquad (3.28)$$

$$X^T_{i,i-1} J_{(t^2)} = J_{(t^2)} ; \quad J_{(t^2)} X^T_{i,i-1} = J_{(t^2)} .$$

5. $$J_{(t^2)} N = (2t-1)J_{(t^2)} ; \quad NJ_{(t^2)} = (2t-1) J_{(t^2)}$$

$$N^T J_{(t^2)} = (2t-1)J_{(t^2)} ; \quad J_{(t^2)} N^T = (2t-1) J_{(t^2)} \qquad (3.29)$$

$$R_0 N = (2t-1) R_0 ; \quad NR_0 = (2t-1) R_0 .$$

6. $$J_{(t^2)} X^T_{10} = J_{(t^2)} ; \quad X^T_{10} J_{(t^2)} = J_{(t^2)} \qquad (3.30)$$

$$X_{10} J_{(t^2)} = J_{(t^2)} ; \quad J_{(t^2)} X_{10} = J_{(t^2)} .$$

7. $$R_0 X^T_{10} = R_0 ; \quad X^T_{10} R_0 = R_0 \qquad (3.31)$$

$$X_{10} R_0 = R_0 ; \quad R_0 X_{10} = R_0 .$$

8. $$R_1 X^T_{10} = X^T_{10} . \qquad (3.32)$$

9. $$R_2 X^T_{10} = R_0 . \qquad (3.33)$$

10. $$X^T_{10} X_{10} = R_1 ; \quad X^T_{01} X_{01} = \Pi R_1 \Pi \neq I_{(t^2)} . \qquad (3.34)$$

While some of these relationships are obvious, the others are proved in Appendix 3.2 at the end of this chapter.

(II). Each treatment level occurs twice with each subject. Hence

$$\sum_1 X^T_i = 2 J_{(t \times t^2)} . \qquad (3.35)$$

3.4.2 Combinations in periods i, i'

In some designs we need to check the combinations in periods more than one apart. A generalisation of the condition in subsection 3.4.1 is $X_i^T X_{i'} = J_{(t)}$. This ensures that all combinations occur in periods i and i'. This condition may appear in other guises. For example, if $X_i^T X_{i'} = J_{(t)}$, then

$$(X_i^T \otimes 1_{(t)})(X_{i'} \otimes 1_{(t)}^T) = J_{(t^2)}$$

$$\text{or} \quad (1_{(t)} \otimes X_i^T)(1_{(t)}^T \otimes X_{i'}) = J_{(t^2)} .$$

$$\text{Also} \quad (X_{i'} \otimes 1_{(t)}^T)\,\Pi = 1_{(t)}^T \otimes X_{i'} .$$

$$\text{Hence} \quad (X_i^T \otimes 1_{(t)})(1_{(t)}^T \otimes X_{i'}) = J_{(t^2)} .$$

In particular, the condition for periods 1, 2t to include all combinations is

$$X_{2t}^T X_1 = J_{(t)} . \tag{3.36}$$

3.4.3 Information matrix for SF designs

Using equations (3.24), (3.34), equation (3.23) for the information matrix eliminating subjects simplifies to

$$C = R_1 + (2t-1)I_{(t^2)} - \frac{1}{2t}(MM^T) , \tag{3.37}$$

where $\quad M = N_0 + N = \sum_1 X_{i,i-1}^T ,$

and $\quad X_{10} = \frac{1}{t}(X_1 \otimes 1_{(t)}^T) .$

Note that equation (3.37) applies only to SF designs.

3.4.4 Order of subjects

Further simplification is possible when a standard order of subjects is used. We consider two such orders, A and B .

Order A : The following order of subjects is used.

	Subject				
	1 2 3 ... t	t+1	...	$t(l_1-1)+l_2$	... t^2
Period	Levels				
1	1 1 1 ... 1	2	...	l_1	... t
2	1 2 3 ... t	1	...	l_2	... t

Then we have,

$$X_{12} = I_{(t^2)} \; .$$

$$X_1 = X_{12}Q_1 = Q_1 \; .$$

$$X_2 = Q_2 \; .$$

$$X_{10} = \frac{1}{t}\,(Q_1 \otimes 1^T_{(t)}) = R_1 \; . \qquad (3.38)$$

And (3.36) becomes

$$X^T_{2t}X_1 = X^T_{2t}Q_1 = J_{(t)} \; . \qquad (3.39)$$

Now $tR_1 = Q_1 \otimes 1^T_{(t)}$.

Hence (3.39) becomes

$$(X^T_{2t} \otimes 1_{(t)})\, R_1 = t^{-1}\, J_{(t^2)} \; . \qquad (3.40)$$

Proof:

$$(X^T_{2t} \otimes 1_{(t)})\, R_1 = (X^T_{2t} \otimes 1_{(t)})\,\frac{1}{t}\,(Q_1 \otimes 1^T_{(t)})$$

$$= \frac{1}{t}\,(X^T_{2t}Q_1) \otimes J_{(t)}$$

$$= \frac{1}{t}\, J_{(t)} \otimes J_{(t)}$$

$$= t^{-1}\, J_{(t^2)} \; .$$

Order B : The following order of subjects is used.

	Subject										
	1	2	3	...	t	t+1	...	$t(l_2-1)+l_1$	...	t^2
Period	Levels									
1	1	2	3	...	t	1	...	l_1	...	t
2	1	1	1	...	1	2	...	l_2	...	t

Then we have,

$$X_{21} = I_{(t^2)} \ .$$

$$X_1 = X_{21}Q_2 = Q_2 \ .$$

$$X_2 = Q_1 \ .$$

$$X_{01} = \frac{1}{t}\,(1^T_{(t)} \otimes Q_2) = R_2 \ .$$

$$X_{10} = R_2\,\Pi$$

3.4.5 Some algebraic results for SF designs.

In this subsection, we consider some algebraic results for the SF designs. These results are useful for the remaining sections of this chapter. They are

I. $$R_1N = 2t^{-1}\,J_{(t^2)} - X^T_{10} \ . \qquad (3.41)$$

When subjects are arranged in Order A (see subsection 3.4.4), equation (3.41) becomes

$$R_1N = 2t^{-1}\,J_{(t^2)} - R_1 \ .$$

II. $$R_2N = 2t^{-1}\,J_{(t^2)} - \frac{1}{t}\,1_{(t)} \otimes X^T_{2t} \ . \qquad (3.42)$$

III. $$R_1NN^TR_1 = \left\{ I_{(t)} + 4(t-1)\,J_{(t)} \right\} \otimes \frac{1}{t}\,J_{(t)} \ . \qquad (3.43)$$

IV. $$R_2NN^TR_2 = \frac{1}{t}\,J_{(t)} \otimes \left\{ I_{(t)} + 4(t-1)\,J_{(t)} \right\} \ . \qquad (3.44)$$

V a. $T_1(X_{10}^T + N) = 0$. (3.45)

V b. $T_1NN^TT_1 = T_1$. (3.46)

VI a. $T_2NN^TT_2 = T_2$. (3.47)

VI b. $T_2N \neq 0$. (3.48)

The proofs of all the above algebraic results are given in Appendix 3.3 at the end of the chapter.

3.5 Some properties of SF designs.

In this section, we consider some statistical properties of SF designs. These properties are given in the four theorems below.

Theorem 3.1 In all SF designs, direct effects (unadjusted for residual effects and interaction) are orthogonal to subject differences.

Proof: Premultiply the normal equation (3.19) by T_1 . The term in $\hat{\beta}$ is $T_1(N_0 + N)\hat{\beta}$. But $N_0 = X_{10}^T$. Also, from (3.45), $T_1(X_{10}^T + N) = 0$ for all SF designs. Therefore, the term in $\hat{\beta}$ vanishes.

Alternative proof: Here we show directly that $T_1CT_1 = T_1C_0T_1$.

where $C_0 = X_{10}^T X_{10} + \sum_2 X_{i,i-1}^T X_{i,i-1}$,

and $C = C_0 - \frac{1}{p}(N_0 + N)(N_0^T + N^T)$.

Now $T_1C_0T_1 = T_1X_{10}^T X_{10} T_1 + T_1 \sum_2 X_{i,i-1}^T X_{i,i-1} T_1$,

$$T_1CT_1 = T_1C_0T_1 - \frac{1}{p}T_1(N_0 + N)(N_0^T + N^T)T_1 ,$$

but $T_1(N_0 + N) = 0$ from (3.45)

so that $T_1(N_0 + N)(N_0^T + N^T)T_1 = 0$.

Hence $T_1CT_1 = T_1C_0T_1$.

Theorem 3.2 In all SF designs, residual effects are not orthogonal to subject differences.

Proof: We require to show $T_2(N_0 + N) \neq 0$.

Now $T_2N_0 = R_2N_0 - R_0N_0$

$$= R_2X_{10}^T - R_0X_{10}^T$$

$$= R_0 - R_0 \qquad \text{from (3.33) and (3.31)}$$

$$= 0 .$$

But $T_2N \neq 0$. from (3.47)

Hence $T_2(N_0 + N) \neq 0$.

Theorem 3.3 Let $U\alpha$ be a vector of effects such that $UT_0 = 0$, $UT_1 = 0$ and NN^T is non-singular. Then in all SF designs, the elements of $U\alpha$ are not orthogonal to subject differences.

Proof: We have to show $U(N_0 + N)$ is not zero. Now U can be expressed as $I_{(t^2)} - T_0 - T_1$.

Therefore,

$$U(N_0 + N) = (I_{(t^2)} - T_0 - T_1)(N_0 + N)$$

$$= (N_0 + N) - T_0(N_0 + N) - T_1(N_0 + N) .$$

But $T_1(N_0 + N) = 0$, from (3.45)

and $T_0(N_0 + N) = R_0X_{10}^T + R_0N$

$$= R_0 + (2t-1)\,R_0 \qquad \text{from (3.31) and (3.29)}$$

$$= 2tR_0 .$$

Hence $U(N_0 + N) = N_0 + N - 2tR_0$

$\neq 0$ if $N \neq -(N_0 - 2tR_0)$, a singular matrix.

<u>Theorem 3.4</u> In all SF designs, direct effects are orthogonal to effects $U\alpha$, even after adjustment for subject differences, where U defined as in previous theorem. In particular, adjusted direct effects are always orthogonal to first residual effects and direct × first residual interaction.

Proof: We require to show rows of T_1 are latent vectors of C .

$$\text{Now } T_1C = T_1\left\{X_{10}^T X_{10} + \sum_2 X_{i,i-1}^T X_{i,i-1} - \frac{1}{p}(N_0 + N)(N_0^T + N^T)\right\} ,$$

$$\text{but } T_1X_{10}^T X_{10} = T_1R_1 = (R_1 - R_0)R_1 = R_1 - R_0$$

$$= T_1 ,$$

$$T_1\left(\sum_2 X_{i,i-1}^T X_{i,i-1}\right) = T_1\,(2t-1)\,I_{(t^2)}$$

$$= (2t-1)\,T_1 ,$$

$$T_1(N_0 + N) = 0 . \qquad \text{from (3.45)}$$

Therefore $T_1C = 2t\,T_1$.

Hence the rows of T_1 are latent vectors of C and $2t$ is a latent root with multiplicity $t-1$.

Corollary 1. : Theorems 3.1 and 3.4 mean that in all SF designs, direct effects, adjusted for effects $U\alpha$, defined in Theorem 3.3, are orthogonal to subject differences.

Corollary 2. : Theorems 3.3 and 3.4 mean that any contrast, other than direct effects, cannot be orthogonal to subject differences.

<u>3.6 Types of SF designs</u>

In this section, we consider two special types of SF designs which will be studied in some details in section 3.8 of this chapter, and Chapter Five. They are R-orthogonal SF designs and binary SF designs.

3.6.1 R-orthogonal SF designs.

A design is described as R-orthogonal (R-ortho in short) if residual effects, adjusted for subjects, are orthogonal to direct × residual interaction as well as direct effects. The general condition for R-ortho designs is $T_2 C T_{12} = 0$. Alternatively, a design is R-ortho if and only if the rows of T_2 are latent vectors of C . Theorem 3.5 below sets out the conditions for R-orthogonality of SF designs. Condition (1) and the interpretation of condition (2) given in the next subsection enable us to ascertain whether a design is R-ortho by merely examining the treatments in the design.

Theorem 3.5 A SF design is R-ortho when

(1) all combinations of treatment levels occur in periods 1 and 2t .

(2) $T_2 N N^T = T_2$.

Proof: Condition (1) is equivalent to $X_{2t}^T X_1 = J_{(t)}$

Now

$$T_2 C = T_2 \left\{ X_{10}^T X_{10} + \sum_2 X_{i,i-1}^T X_{i,i-1} - \frac{1}{p} (N_0 + N)(N_0^T + N^T) \right\} .$$

But $T_2 X_{10}^T X_{10} = T_2 R_1 = 0$,

$$T_2 (\sum_2 X_{i,i-1}^T X_{i,i-1}) = T_2 \, (2t-1) \, I_{(t^2)}$$

$$= (2t-1) \, T_2 \quad ,$$

$T_2 N_0 N_0^T = 0$ since, from Theorem 3.2, $T_2 N_0 = 0$.

$T_2 N_0 N^T = 0$ since $T_2 N_0 = 0$.

$T_2 N N^T = T_2$ from condition (2) .

Now $T_2 N = R_2 N - R_0 N$

$$= \frac{2}{t} J_{(t^2)} - \frac{1}{t} 1_{(t)} \otimes X_{2t}^T - \frac{(2t-1)}{t} J_{(t^2)} \qquad \text{from (3.42) and (3.29)}$$

$$= \frac{1}{t^2} J_{(t^2)} - \frac{1}{t} 1_{(t)} \otimes X_{2t}^T \quad .$$

$$\therefore \; T_2NN_0^T = \frac{1}{t^2} J_{(t^2)} X_{10} - \frac{1}{t}(1_{(t)} \otimes X_{2t}^T) X_{10}$$

$$= \frac{1}{t^2} J_{(t^2)} - \frac{1}{t}(1_{(t)} \otimes X_{2t}^T) \frac{1}{t}(1_{(t)}^T \otimes X_1)\Pi \quad \text{from (3.16)}$$

$$= \frac{1}{t^2} J_{(t^2)} - \frac{1}{t^2}\left\{ J_{(t)} \otimes (X_{2t}^T X_1) \right\}\Pi$$

$$= \frac{1}{t^2} J_{(t^2)} - \frac{1}{t^2}\left\{ J_{(t)} \otimes J_{(t)} \right\}\Pi \;, \quad \text{from condition (1)}$$

$$= \frac{1}{t^2} J_{(t^2)} - \frac{1}{t^2} J_{(t^2)}\Pi$$

$$= 0 \;.$$

Hence $T_2C = (2t-1-\frac{1}{2t})\, T_2$.

Note: (i) When the conditions are met, $2t-1-\dfrac{1}{2t}$ is a latent root of C with multiplicity $t-1$.

(ii) The conditions are shown here to be sufficient. They are not shown to be necessary.

3.6.2 Interpretation of the condition $T_2NN^T = T_2$ for R-orthogonality .

In this subsection, we examine the interpretation of the second condition for R-orthogonality of SF designs in subsection 3.6.1 , that is, $T_2NN^T = T_2$.

Now $\quad T_2NN^T = T_2$.

But $\quad T_2N = \frac{1}{t^2} J_{(t^2)} - \frac{1}{t} 1_{(t)} \otimes X_{2t}^T$. $\quad$ see subsection 3.6.1

Therefore, $T_2NN^T = \dfrac{(2t-1)}{t^2} J_{(t^2)} - \dfrac{1}{t}(1_{(t)} \otimes X_{2t}^T)\, N^T$ $\quad$ from (3.29)

and $T_2 = R_2 - R_0 = \frac{1}{t} J_{(t)} \otimes I_{(t)} - \frac{1}{t^2} J_{(t^2)}$.

$$\therefore \; (1_{(t)} \otimes X_{2t}^T)\, N^T = -t R_2 + 2J_{(t^2)} \;.$$

but $(1_{(t)} \otimes X_{2t}^T)\, N^T = 1_{(t)} \otimes (X_{2t}^T N^T)$,

$$\text{and} \quad - t R_2 + 2 J_{(t^2)} = - J_{(t)} \otimes I_{(t)} + 2 J_{(t)} \otimes J_{(t)}$$

$$= J_{(t)} \otimes (2 J_{(t)} - I_{(t)})$$

$$= 1_{(t)} \otimes \left\{ 1^T_{(t)} \otimes (2 J_{(t)} - I_{(t)}) \right\} .$$

Hence $1_{(t)} \otimes (X^T_{2t} N^T) = 1_{(t)} \otimes \left\{ 1^T_{(t)} \otimes (2 J_{(t)} - I_{(t)}) \right\}$,

where cell (l, v) of $\underset{(t \times t^2)}{X^T_{2t} N^T}$ gives the number of times combination

v occurs with subjects that have treatment level l in period $2t$.

The frequencies should be as follows:

	frequency
Combination $t(l'-1) + l$, $l' = 1, 2, \ldots, t$	1
where l, l' are levels in periods $i, i-1$	
Others	2

For example, it can be shown that the design of Table 2.8 is R-orthogonal since it satisfies condition (1) of Theorem 3.5 and the above frequencies required for R-orthogonality. The design in Table 4.14 in Chapter Four satisfies condition (1) of Theorem 3.5 but does not have the required frequencies and therefore does not satisfy condition (2). Thus, for subjects with treatment level 1 in period 2t , the frequencies for this design are:

Combination	Frequency	Required frequency
1	1	1
4	0	1
7	2	1

3.6.3 Binary SF designs.

An incomplete block design with all elements of the incidence matrix, N , either 0 or 1 is called binary. It follows that all diagonal elements of its concurrence matrix, NN^T, are equal. In particular, an SF design in which its associated incomplete block design is binary will be called a binary SF design. For example, the design by Berenblut (1964) in Table 2.8 is not binary as can be observed from the incidence matrix, N, of its a.i.b. design in Table 3.2. The importance of this property will be shown in subsection 3.8.1 of this chapter.

3.7 Balance

In this section, we examine the question of balance in SF designs with respect to direct effects, residual effects and direct × residual interaction.

A design is balanced for a set of orthogonal normalised effects $L\alpha$, $LL^T = I$, when the estimated effects all have the same variance.

The following three theorems hold for SF designs.

Theorem 3.6 : All SF designs are balanced for any set of $(t-1)$ orthogonal normalised direct effect contrasts $L\alpha$ where $LL^T = I_{(t-1)}$, $L^TL = T_1$.

Proof: From Theorem 3.4, we have for all SF designs

$$T_1C = 2tT_1 ,$$

$$\text{that is,} \quad T_1CT_1 = 2tT_1 .$$

$$\begin{aligned} \therefore \quad LCL^T &= (LL^T)LCL^T(LL^T) \\ &= LT_1CT_1L^T \\ &= 2tLT_1L^T \\ &= 2tLL^TLL^T \\ &= 2tI_{(t-1)} . \end{aligned}$$

$$\therefore (LCL^T)^{-1} = \frac{1}{2t} I_{(t-1)} .$$

Hence the estimated effects $L\alpha$ all have the same variance $\sigma^2/2t$ and

and are uncorrelated with one another.

<u>Theorem 3.7</u>: All SF designs are balanced for any set of $(t-1)$ orthogonal normalised residual effect contrasts $U\alpha$, $UU^T = I_{(t-1)}$, $U^TU = T_2$ provided they satisfy the conditions for R-orthogonality in Theorem 3.5.

Proof: If the conditions for R-ortho designs in Theorem 3.5 are met, then

$$T_2C = (2t-1-\frac{1}{2t})\,T_2$$

$$\therefore \quad T_2C = T_2CT_2$$

$$\begin{aligned}\therefore \quad UCU^T &= (UU^T)\,UCU^T\,(UU^T)\\ &= UT_2CT_2U^T\\ &= (2t-1-\frac{1}{2t})\,UT_2U^T\\ &= (2t-1-\frac{1}{2t})\,I_{(t-1)}\end{aligned}$$

$$\therefore (UCU^T)^{-1} = \frac{1}{(2t-1-\frac{1}{2t})}\,I_{(t-1)}$$

Hence the estimated effects $U\alpha$ all have the same variance $\sigma^2/(2t-1-\frac{1}{2t})$.

<u>Theorem 3.8</u>: No SF design with $t > 3$ is balanced for all direct × first residual interactions.

Proof: To get balance, we require the spectral decomposition of NN^T to be

$$NN^T = (\frac{2t-1}{t})^2 J_{(t^2)} + T_1 + T_2 + \theta\,(I_{(t^2)} - T_1 - T_2 - \frac{1}{t^2} J_{(t^2)})$$

where θ is scalar and $R_0NN^TR_0 = (2t-1)^2R_0$. The decomposition given above is possible only when NN^T is of the form

$$\begin{bmatrix} a & b & b \cdots b & c & c \cdots b & c & c \cdots \\ b & a & b \cdots c & b & c \cdots c & b & c \cdots \\ b & b & a \cdots c & c & b \cdots c & c & b \cdots \\ \vdots & \vdots & \vdots & & \vdots & & & & \\ b & c & c & a & b & b & & & \\ c & b & c & b & a & b & & & \\ c & c & b & b & b & a & & & \\ & \vdots & & & \vdots & & & & \end{bmatrix}$$

as $I_{(t^2)}$, $J_{(t^2)}$, $T_1 + T_2$ are all of this form. (Cf. Pearce (1963)). We require integral values of a, b, c. Since the design is binary, a must be $(2t-1)$. Also

$$R_1 NN^T R_1 = \left\{ I_{(t)} + 4(t-1) J_{(t)} \right\} \otimes \frac{1}{t} J_{(t)} .$$

$$R_2 NN^T R_2 = \frac{1}{t} J_{(t)} \otimes \left\{ I_{(t)} + 4(t-1) J_{(t)} \right\} .$$

Hence $a + (t-1) b = 4t-3$

$b + (t-1) c = 4t-4$

$\therefore \quad b = 2, c = 4 - 2/(t-1)$.

c is an integer only when $t = 2, 3$.

Note: There is only one SF design with 2 treatments. This design has the above balance. No design for 3 treatments has been found with this balance (see Chapter Four).

In Theorem 3.9 below, we show that no SF design with $t > 2$ exists such that its a.i.b. design is a balanced incomplete block (b.i.b.) design, that is, $NN^T = k_1 I + k_2 J$ for some constants k_1, k_2 satisfying the results of the combinatorial properties of SF designs given in subsection 3.4.5. This is important because if the a.i.b. design of ~~a~~ an SF design is a b.i.b. design, then the SF design is balanced for any set of orthogonal normalised contrasts of residual effects and direct × residual interaction, and the contrasts are uncorrelated with one another, as shown in Theorem 3.10. (Such a design would be R-orthogonal).

Theorem 3.9: No SF design with $t > 2$ exists such that its a.i.b. design is a balanced incomplete block design. That is, there is no SF design with $t > 2$ such that $NN^T = k_1 I + k_2 J$ where k_1, k_2 are non-negative integers.

Proof: We require integral values of k_1, k_2. Since the design is binary, $k_1 + k_2 = 2t - 1$.

Now

$$R_1 NN^T R_1 = \left\{ I_{(t)} + 4(t-1)\, J_{(t)} \right\} \otimes \frac{1}{t} J_{(t)} \,.$$

$$R_2 NN^T R_2 = \frac{1}{t} J_{(t)} \otimes \left\{ I_{(t)} + 4(t-1)\, J_{(t)} \right\} .$$

Hence $(k_1 + k_2) + (t-1)k_2 = 4t-3$

$$tk_2 = 4t-4$$

$\therefore \quad k_1 = 1 \,, \; k_2 = 4 - 4/t$. But $k_1 + k_2 = 2t-1$.

$\therefore$ k_2 is an integer only when $t = 2$.

<u>Theorem 3.10</u> : A SF design is balanced for any set of orthogonal normalised contrasts of residual effects and direct × residual interaction, $W\alpha$, where $WW^T = I_{(t^2-t)}$, $W^TW = U = I_{(t^2)} - T_0 - T_1$, and the contrasts are uncorrelated with one another, if its a.i.b. design is a balanced incomplete block design.

Proof: The proof will be in two parts.

(1) First we show

$$(I_{(t^2)} - T_0 - T_1)\, C\, (I_{(t^2)} - T_0 - T_1) = (2t-1-\frac{k_1}{2t})(I_{(t^2)} - T_0 - T_1)$$

where $C = R_1 + (2t-1)\, I_{(t^2)} - \frac{1}{2t}(N_0 + N)(N_0^T + N^T)$,

$$NN^T = k_1\, I_{(t^2)} + k_2\, J_{(t^2)} \quad .$$

Now

$$UR_1U^T = (I_{(t^2)} - T_0 - T_1)\, R_1\, (I_{(t^2)} - T_0 - T_1)$$

$$= 0 \quad .$$

$$U\,(2t-1)\, I_{(t^2)}\, U^T = (2t-1)\, U \qquad \text{since } UU^T = U \;.$$

$$UN_0N^TU^T = (I_{(t^2)} - T_0 - T_1)\, X_{10}^T\, N^TU^T$$

$$= (X_{10}^T - X_{10}^T)\, N^TU^T$$

$$= 0 \;.$$

Similarly, $UNN_0^T U^T = 0$.

Also $$UNN^T U^T = U\,(k_1\, I_{(t^2)} + k_2\, J_{(t^2)})\, U^T$$

$$= k_1\, U \; .$$

Hence the result.

(2) $$WCW^T = (WW^T)\, WCW^T\, (WW^T)$$

$$= WUCUW^T$$

$$= (2t-1- {}^{k_1}/_{2t})\, WUW^T$$

$$= (2t-1- {}^{k_1}/_{2t})\, I_{(t^2-t)} \; .$$

Hence the estimated effects $W\alpha$ all have the same variance $\sigma^2/(2t-1- {}^{k_1}/_{2t})$ and are uncorrelated with one another.

Corollary: No SF design with $t > 2$ exists such that it is balanced for any set of orthogonal normalised contrasts of residual effects and direct × residual interaction, and the contrasts are uncorrelated with one another.

3.8 Within-subject information in R-ortho SF designs.

In this section, we consider the subdivision of the total information (adjusted and unadjusted for subjects) into information on the mean, direct effects, residual effects and direct × residual interaction. Now, a measure of the total information is

tr C (adjusted for subjects),

or tr C_0 (unadjusted for subjects),

where tr denotes trace

and $$C = R_1 + (2t-1)\, I_{(t^2)} - \frac{1}{2t}\,(N_0 + N)(N_0^T + N^T) \; .$$

$$C_0 = R_1 + (2t-1)\, I_{(t^2)} \; .$$

The subdivision of information is as follows:

	Adjusted	Not adjusted
Mean	tr R_0CR_0	tr $R_0C_0R_0$
Direct Effects	tr T_1CT_1	tr $T_1C_0T_1$
Residual Effects	tr T_2CT_2	tr $T_2C_0T_2$
Interaction	(by difference)	(by difference)
Total	tr C	tr C_0

But we have

1. $\operatorname{tr} R_0C_0R_0 = 2t$.
2. $\operatorname{tr} R_0CR_0 = 0$.
3. $\operatorname{tr} T_1C_0T_1 = 2t(t-1)$.
4. $\operatorname{tr} T_1CT_1 = 2t(t-1)$.
5. $\operatorname{tr} T_2C_0T_2 = (2t-1)(t-1)$.
6. $\operatorname{tr} T_2CT_2 = (2t-1)(t-1)\left\{1 - \frac{1}{2t(2t-1)}\right\}$.
7. $\operatorname{tr} C_0 = t + t^2(2t-1)$.
8. $\operatorname{tr} C = t + t^2(2t-1) - \frac{1}{2t}(3t + \operatorname{tr} NN^T)$.

These results are proved in Appendix 3.4 at the end of the chapter.

Therefore, we have the following subdivision of information:

	Adjusted	Unadjusted
Mean	0	$2t$
Direct Effects	$2t(t-1)$	$2t(t-1)$
Residual Effects	$(2t-1)(t-1)\left\{1 - \frac{1}{2t(2t-1)}\right\}$	$(2t-1)(t-1$
Interaction	x	$x_0 = (2t-1)(t-1)^2$
Total	$t + t^2(2t-1) - \frac{1}{2t}(3t + \operatorname{tr} NN^T)$	$t + t^2(2t-1)$

$$x = (2t-1)(t-1)^2\left\{1 + \frac{1}{(t-1)^2} - \frac{1 + \operatorname{tr} NN^T}{2t(t-1)^2(2t-1)}\right\} .$$

3.8.1 Fractional loss of information on residual effects and direct × residual interaction.

The fractional loss of information on residual effects is $\frac{1}{2t(2t-1)}$ for all R-ortho SF designs. For example, for the design by Berenblut (1964) in Table 2.8, the value is $1/30$.

The fractional loss of information on interaction is

$$\frac{x_0 - x}{x_0} = f ,$$

given by

$$f = \frac{1}{2t(t-1)^2(2t-1)} \left\{ 1 + \mathrm{tr}\, NN^T - 2t(2t-1) \right\} .$$

Value of f for binary SF designs

For a binary design, $\mathrm{tr}\, NN^T = t^2(2t-1)$. Therefore $f = f_1$ say $= \frac{1}{2t} - \frac{1}{t(t-1)(2t-1)}$.

For example, when $t = 3$, $f_1 = \frac{1}{6} - \frac{1}{30} = 2/15$.

Value of f for non-binary SF designs

If m_1 combinations occur twice with one subject,
m_2 combinations occur twice with two subjects,
etc.

then

$$f = f_1 + \frac{m_1 + 2m_2 + 3m_3 + \dots}{t(t-1)^2(2t-1)} .$$

For example, the value of f for the design by Berenblut (1964) in Table 2.8 is

$$f = \frac{2}{15} + \frac{2.3}{3.4.5} = 7/30 .$$

Thus we find that binary designs have minimum loss of information on interaction. This property of binary design is useful in Chapter Five.

Appendix 3.1 . Some special matrix definitions.

We define the following:

$$R_0 = \frac{1}{t} J_{(t)} \otimes \frac{1}{t} J_{(t)} \quad .$$

$$R_1 = I_{(t)} \otimes \frac{1}{t} J_{(t)} = \frac{1}{t} Q_1 \otimes 1^T_{(t)} \quad . \qquad (3.17)$$

$$R_2 = \frac{1}{t} J_{(t)} \otimes I_{(t)} = \frac{1}{t} I^T_{(t)} \otimes Q_2 \quad .$$

$$R_{12} = I_{(t^2)} \quad .$$

where R_0, R_1, R_2, R_{12} are idempotent and symmetric matrices, and $R_0R_1 = R_0R_2 = R_1R_2 = R_0$.

Let $T_0 = R_0$.

$$T_1 = R_1 - R_0 \quad . \qquad (3.18)$$

$$T_2 = R_2 - R_0 \quad .$$

$$T_{12} = R_{12} - R_1 - R_2 + R_0 \quad ,$$

where T_0, T_1, T_2, T_{12} are idempotent and symmetric matrices such that all matrix products of them are zero. Also $T_0 + T_1 + T_2 + T_{12} = I_{(t^2)}$. Then we have the following operations. (Cf : Kurkjian and Zelen (1962, 1963)).

$R_0\alpha$	replaces	each	element	of	α	by its mean
$R_1\alpha$	"	"	"	"	"	by row mean
$R_2\alpha$	"	"	"	"	"	by column mean
$T_1\alpha$	"	"	"	"	"	by deviation of row means from overall mean
$T_2\alpha$	"	"	"	"	"	by deviation of column means from overall mean
$T_{12}\alpha$	"	"	"	"	"	by second difference $\alpha_{lm} - \bar{\alpha}_{l.} - \bar{\alpha}_{.m} + \bar{\alpha}_{..}$

where $\alpha = (\alpha_{11}\ \alpha_{12} \ldots \alpha_{1t}\ \alpha_{21} \ldots \alpha_{lm} \ldots \alpha_{tt})^T$ and α_{lm} is the treatment combination mean with treatment level l in period i and treatment level m in period i-1 . The l^{th} row mean, $\bar{\alpha}_{l.}$, the m^{th} column mean $\bar{\alpha}_{.m}$ and the general mean, $\bar{\alpha}_{..}$, are obtained below.

$$
\begin{array}{llllll}
 & & & & \text{Row Total} & \text{Row Mean} \\
 & \alpha_{11} \cdots & \alpha_{1m} \cdots & \alpha_{1t} & \alpha_{1.} & \bar{\alpha}_{1.} \\
 & \vdots & & \vdots & & \\
 & \alpha_{l1} \cdots & \alpha_{lm} \cdots & \alpha_{lt} & \alpha_{l.} & \bar{\alpha}_{l.} \\
 & \vdots & & \vdots & & \\
 & \alpha_{t1} \cdots & \alpha_{tm} \cdots & \alpha_{tt} & \alpha_{t.} & \bar{\alpha}_{t.} \\
\text{Column total} & \alpha_{.1} & \alpha_{.m} & \alpha_{.t} & & \\
\text{Column mean} & \bar{\alpha}_{.1} & \bar{\alpha}_{.m} & \bar{\alpha}_{.t} & & \bar{\alpha}_{..} = \sum_{l=1}^{t} \sum_{m=1}^{t} \alpha_{ij}/t^2
\end{array}
$$

Appendix 3.2 . Proofs of results in subsection 3.4.1

3. $J_{(t^2)}(1^T_{(t)} \otimes X_i) = t\, J_{(t^2)}$. (3.27)

Proof:
$$
\begin{aligned}
J_{(t^2)}(1^T_{(t)} \otimes X_i) &= (1_{(t^2)} \otimes 1^T_{(t^2)})(1^T_{(t)} \otimes X_i) \\
&= \underset{(t^2 \times t)}{J} \otimes (1^T_{(t^2)} X_i) \\
&= \underset{(t^2 \times t)}{J} \otimes t\, 1^T_{(t)} \\
&= t\, J_{(t^2)} .
\end{aligned}
$$

6a. $J_{(t^2)} X^T_{10} = J_{(t^2)}$.

Proof: $J_{(t^2)} X^T_{10}$
$$
\begin{aligned}
&= (J_{(t)} \otimes J_{(t)}) \tfrac{1}{t} (X^T_1 \otimes 1_{(t)}) \\
&= \tfrac{1}{t} \underset{(t \times t^2)}{J} \otimes t\, 1_{(t)} \\
&= J_{(t^2)} .
\end{aligned}
$$

6b. $X^T_{10} J_{(t^2)} = J_{(t^2)}$. (3.30)

Proof: $X^T_{10} J_{(t^2)}$
$$
\begin{aligned}
&= \tfrac{1}{t} (X^T_1 \otimes 1_{(t)})(1_{(t^2)} \otimes 1^T_{(t^2)}) \\
&= \tfrac{1}{t}\, t\, 1_{(t)} \otimes \underset{(t \times t^2)}{J} \\
&= J_{(t^2)} .
\end{aligned}
$$

8. $R_1 X_{10}^T = X_{10}^T$. (3.32)

Proof: $R_1 X_{10}^T = (I_{(t)} \otimes \frac{1}{t} J_{(t)}) \frac{1}{t} (X_1^T \otimes 1_{(t)})$

$$= \frac{1}{t} X_1^T \otimes 1_{(t)}$$

$$= X_{10}^T .$$

9. $R_2 X_{10}^T = R_0$. (3.33)

Proof: $R_2 X_{10}^T = (\frac{1}{t} J_{(t)} \otimes I_{(t)}) \frac{1}{t} (X_1^T \otimes 1_{(t)})$

$$= \frac{1}{t^2} J_{(t \times t^2)} \otimes 1_{(t)}$$

$$= \frac{1}{t^2} J_{(t^2)}$$

$$= R_0 .$$

10. $X_{10}^T X_{10} = R_1$. (3.34)

Proof: $X_{10}^T X_{10}^T = \frac{1}{t} (X_1^T \otimes 1_{(t)}) \frac{1}{t} (X_1 \otimes 1_{(t)}^T)$

$$= \frac{1}{t^2} (X_1^T X_1) \otimes J_{(t)}$$

$$= \frac{1}{t^2} \; t I_{(t)} \otimes J_{(t)}$$

$$= R_1 .$$

Appendix 3.3. Proofs of results in subsection 3.4.5

I. $R_1 N = 2t^{-1} J_{(t^2)} - X_{10}^T$. (3.41)

Proof: $R_1 N = R_1 (X_{21}^T + X_{32}^T + \dots + X_{2t,2t-1}^T)$

$= \frac{1}{t} (X_2^T + \dots + X_{2t}^T) \otimes 1_{(t)}$ from (3.13)

$= \frac{1}{t} (2 J_{(t \times t^2)} - X_1^T) \otimes 1_{(t)}$ from (3.35)

$= 2t^{-1} J_{(t^2)} - X_{10}^T$. from (3.15)

II. $R_2N = 2t^{-1}\, J_{(t^2)} - \frac{1}{t}\, 1_{(t)} \otimes X_{2t}^T$. (3.42)

Proof: $R_2N = R_2(X_{21}^T + X_{32}^T + \dots + X_{2t,2t-1}^T)$

$= \frac{1}{t}\, 1_{(t)} \otimes (X_1^T + X_2^T + \dots + X_{2t-1}^T)$ from (3.12)

$= \frac{1}{t}\, 1_{(t)} \otimes (2\, J_{(t\times t^2)} - X_{2t}^T)$ from (3.35)

$= 2t^{-1}\, J_{(t^2)} - \frac{1}{t}\, 1_{(t)} \otimes X_{2t}^T$.

III. $R_1NN^T R_1 = \left\{ I_{(t)} + 4(t-1)\, J_{(t)} \right\} \otimes \frac{1}{t}\, J_{(t)}$. (3.43)

Proof: Now $R_1N = 2t^{-1}\, J_{(t^2)} - X_{10}^T$. from (3.41)

$\therefore R_1NN^TR_1 = (2t^{-1}\, J_{(t^2)} - X_{10}^T)(2t^{-1}\, J_{(t^2)} - X_{10})$

$= 4\, J_{(t^2)} - 2t^{-1}\, J_{(t^2)}\, X_{10} - 2t^{-1}\, X_{10}^T\, J_{(t^2)} + X_{10}^T\, X_{10}$

$= 4\, J_{(t^2)} - 2t^{-1}\, J_{(t^2)} - 2t^{-1}\, J_{(t^2)} + R_1$ from (3.30) and (3.34)

$= 4(t-1)\, \frac{1}{t}\, J_{(t)} \otimes J_{(t)} + I_{(t)} \otimes \frac{1}{t}\, J_{(t)}$

$= \left\{ I_{(t)} + 4(t-1)\, J_{(t)} \right\} \otimes \frac{1}{t}\, J_{(t)}$.

IV. $R_2NN^TR_2 = \frac{1}{t}\, J_{(t)} \otimes \left\{ I_{(t)} + 4(t-1)\, J_{(t)} \right\}$. (3.44)

Proof: Now $R_2N = 2t^{-1}\, J_{(t^2)} - \frac{1}{t}\, 1_{(t)} \otimes X_{2t}^T$. from (3.42)

$\therefore R_2NN^TR_2 = (2t^{-1}\, J_{(t^2)} - \frac{1}{t}\, 1_{(t)} \otimes X_{2t}^T)(2t^{-1}\, J_{(t^2)} - \frac{1}{t}\, 1_{(t)}^T \otimes X_{2t})$

$= 4J_{(t^2)} - 2t^{-2} J_{(t^2)} (1_{(t)}^T \otimes X_{2t}) - 2t^{-2}(1_{(t)} \otimes X_{2t}^T)\, J_{(t^2)}$

$+ \frac{1}{t^2}\, (1_{(t)} \otimes X_{2t}^T)(1_{(t)}^T \otimes X_{2t})$

$= 4\, J_{(t^2)} - 2t^{-1}\, J_{(t^2)} - 2t^{-1} J_{(t^2)} + \frac{1}{t}\, J_{(t)} \otimes I_{(t)}$ from (3.27) and (3.25)

$= 4(t-1)\, \frac{1}{t}\, J_{(t)} \otimes J_{(t)} + \frac{1}{t}\, J_{(t)} \otimes I_{(t)}$

$= \frac{1}{t}\, J_{(t)} \otimes \left\{ I_{(t)} + 4(t-1)\, J_{(t)} \right\}$.

V a. $T_1(X_{10}^T + N) = 0$. (3.45)

Proof: $T_1(X_{10}^T + N) = T_1X_{10}^T + T_1N$,

but $T_1X_{10}^T = R_1\, X_{10}^T - R_0X_{10}^T$

$= X_{10}^T - R_0$,

and $T_1N = R_1N - R_0N$

$= 2t^{-1}\, J_{(t^2)} - X_{10}^T - \frac{(2t-1)}{t^2}\, J_{(t^2)}$ from (3.41) and (3.29)

$= R_0 - X_{10}^T$.

$\therefore\ T_1(X_{10}^T + N) = 0$.

V b. $T_1NN^TT_1 = T_1$. (3.46)

Proof: From above, $T_1N = R_0 - X_{10}^T$.

$\therefore\ T_1NN^TT_1 = (R_0 - X_{10}^T)(R_0 - X_{10})$

$= R_0 - R_0\, X_{10} - X_{10}^T\, R_0 + X_{10}^T\, X_{10}$

$= R_0 - R_0 - R_0 + R_1$ from (3.31) and (3.34)

$= T_1$.

VI. $T_2NN^TT_2 = T_2$. (3.47)

Proof: Now $T_2N = R_2N - R_0N$

$$= 2t^{-1}\, J_{(t^2)} - \frac{1}{t}\, 1_{(t)} \otimes X_{2t}^T - \frac{(2t-1)}{t^2}\, J_{(t^2)} \quad \text{from (3.42) and (3.29)}$$

$$= \frac{1}{t^2}\, J_{(t^2)} - \frac{1}{t}\, 1_{(t)} \otimes X_{2t}^T \ .$$

$$\therefore T_2NN^TT_2 = \left(\frac{1}{t^2}\, J_{(t^2)} - \frac{1}{t}\, 1_{(t)} \otimes X_{2t}^T\right)\left(\frac{1}{t^2}\, J_{(t^2)} - \frac{1}{t}\, 1_{(t)}^T \otimes X_{2t}\right)$$

$$= \frac{1}{t^2}\, J_{(t^2)} - \frac{1}{t^3}\, J_{(t^2)}(1_{(t)}^T \otimes X_{2t}) - \frac{1}{t^3}\,(1_{(t)} \otimes X_{2t}^T)\, J_{(t^2)}$$

$$+ \frac{1}{t^2}\,(1_{(t)} \otimes X_{2t}^T)(1_{(t)}^T \otimes X_{2t})$$

$$= \frac{1}{t^2}\, J_{(t^2)} - \frac{1}{t^2}\, J_{(t^2)} - \frac{1}{t^2}\, J_{(t^2)} + \frac{1}{t^2}\, J_{(t)} \otimes t\, I_{(t)} \quad \text{from (3.27) and (3.25)}$$

$$= R_2 - R_0$$

$$= T_2 \ .$$

VII. $T_2N \neq 0$. (3.48)

Proof: Since $T_2NN^TT_2 = T_2$, from (3.45)

$\therefore$ $T_2N \neq 0$.

Appendix 3.4. Proofs of results in section 3.8.

1. $\operatorname{tr} R_0C_0R_0 = 2t$.

Proof: $R_0C_0 = R_0 + (2t-1)\, R_0 = 2t\, R_0$.

$\therefore$ $R_0C_0R_0 = 2t\, R_0$.

$\therefore$ $\operatorname{tr} R_0C_0R_0 = 2t$.

2. $\operatorname{tr} R_0 C R_0 = 0$.

Proof: $R_0 C = 2tR_0 - \frac{1}{2t} R_0 \left\{ N_0 N_0^T + N_0 N^T + N N_0^T + N N^T \right\}$.

But $R_0 N_0 N_0^T = R_0 X_{10}^T X_{10} = R_0 R_1 = R_0$,

$R_0 N_0 N^T = R_0 X_{10}^T N^T = R_0 N^T$ from (3.31)

$= (2t-1) R_0$, from (3.29)

$R_0 N N_0^T = (2t-1) R_0 X_{10}$

$= (2t-1) R_0$,

$R_0 N N^T = (2t-1) R_0 N^T$

$= (2t-1)^2 R_0$.

$\therefore \quad R_0 C = 0$.

Hence $\operatorname{tr} R_0 C R_0 = 0$.

3. $\operatorname{tr} T_1 C_0 T_1 = 2t(t-1)$.

Proof: $T_1 C_0 = T_1 + (2t-1) T_1 = 2t T_1$.

$\therefore \quad T_1 C_0 T_1 = 2t T_1$.

Hence $\operatorname{tr} T_1 C_0 T_1 = 2t(t-1)$.

4. $\operatorname{tr} T_1 C T_1 = 2t(t-1)$.

Proof: From Theorem 3.4, we have

$T_1 C = 2t T_1$.

$\therefore \quad T_1 C T_1 = 2t T_1$.

Hence $\operatorname{tr} T_1 C T_1 = 2t(t-1)$.

5. $\operatorname{tr} T_2 C_O T_2 = (2t-1)(t-1)$.

Proof: $T_2 C_O = (2t-1)\, T_2$.

$\therefore \quad T_2 C_O T_2 = (2t-1)\, T_2$.

Hence $\operatorname{tr} T_2 C_O T_2 = (2t-1)(t-1)$.

6. $\operatorname{tr} T_2 C T_2 = (2t-1)(t-1) \left\{ 1 - \frac{1}{2t(2t-1)} \right\}$.

Proof: From Theorem 3.5, we have for R-ortho designs,

$$T_2 C = (2t-1-\frac{1}{2t})\, T_2 \; .$$

$\therefore \quad T_2 C T_2 = (2t-1-\frac{1}{2t})\, T_2$.

Hence
$$\operatorname{tr} T_2 C T_2 = (2t-1-\frac{1}{2t})(t-1)$$
$$= (2t-1)(t-1) \left\{ 1 - \frac{1}{2t(2t-1)} \right\} .$$

7. $\operatorname{tr} C_O = t + t^2\,(2t-1)$.

Proof:
$$\operatorname{tr} C_O = \operatorname{tr} R_1 + (2t-1) \operatorname{tr} I_{(t^2)}$$
$$= t + t^2\,(2t-1) \; .$$

8. $\operatorname{tr} C = t + t^2\,(2t-1) - \frac{1}{2t}\,(3t + \operatorname{tr} NN^T)$.

Proof: $\operatorname{tr} C = t + t^2\,(2t-1) - \frac{1}{2t} \operatorname{tr} \quad N_O N_O^T + N_O N^T + N N_O^T + N N^T$.

But
$$\operatorname{tr} N_O N_O^T = \operatorname{tr} R_1 = t \; ,$$
$$N_O N^T = X_{10}^T\, N^T$$
$$= \frac{1}{t}\,(X_1^T \,(\times)\, 1_{(t)})\, N^T$$
$$= \frac{1}{t}\,(X_1^T N^T)(\times)\, 1_{(t)} \; .$$

Cell $(1, v)$ of $\underset{(t \times t^2)}{X_1^T N^T}$ gives the number of times combination v occurs with subjects that have treatment level 1 in period 1. For all SF designs, $X_1^T N^T$ is of the form given below,

$$\begin{array}{lc}
\begin{array}{l} \text{level in } i \\ \text{level in } i-1 \end{array} & \begin{array}{ccccccc} 1 & 1\ldots1 & 2 & 2\ldots2 & 3\ldots t-1 & t & t\ldots t \\ 1 & 2\ldots t & 1 & 2\ldots t & 1\ldots t & 1 & 2\ldots t \end{array} \\
\begin{array}{lc} & 1 \\ \text{level in} & 2 \\ \text{period 1} & 3 \\ & \vdots \\ & t \end{array} & \left[\begin{array}{ccccccc} a_1 & a_2\ldots a_t & & & & & \\ & & b_1 & b_2\ldots b_t & & & \\ & & & & & & \\ & & & & & & \\ & & & & & h_1 & h_2\ldots h_t \end{array}\right]
\end{array}$$

where $\sum_{i=1}^{t} a_i = \sum_{i=1}^{t} b_i = \ldots = \sum_{i=1}^{t} h_i = t$.

$\therefore \quad \operatorname{tr} N_0 N^T = \frac{1}{t} t^2 = t$.

Similarly, $\operatorname{tr} N N_0^T = t$.

Hence the result.

Chapter Four

Construction of SF changeover designs

4.1 Introduction

Quenouille (1953) first introduced SF designs with t treatments requiring t^2 subjects and $2t$ periods for $t=2$ and 4. He also gave designs with three treatments requiring $2t$ periods but $2t^2$ subjects. Later, Berenblut (1964) constructed designs for any $t \geq 2$. A design for three treatments is already given in Table 2.8. Quenouille (1953) did not have such a design. Using a different method of construction, Patterson (1970) constructed a more extensive class of designs; this class includes the designs constructed by Berenblut (1964). Although Patterson's method is also general for any t treatments, he was especially concerned with $t=4$. A design with 4 treatments that is not one of Berenblut's design (1964) is given in Table 4.1.

In this chapter, we construct a still more extensive class of designs.

Table 4.1 Design DD (1423)(1324) for four treatments.*

	subject															
Period	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII	XIV	XV	XVI
1	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
2	4	1	2	3	1	4	3	2	2	3	4	1	3	2	1	4
3	4	3	2	1	4	3	2	1	4	3	2	1	4	3	2	1
4	2	3	4	1	3	2	1	4	4	1	2	3	1	4	3	2
5	2	1	4	3	2	1	4	3	2	1	4	3	2	1	4	3
6	3	2	1	4	2	3	4	1	1	4	3	2	4	1	2	3
7	3	4	1	2	3	4	1	2	3	4	1	2	3	4	1	2
8	1	4	3	2	4	1	2	3	3	2	1	4	2	3	4	1

* The notation is explained in section 4.3.

4.2 Method of construction of designs by Berenblut (1964).

In this section, we briefly describe the construction of designs by Berenblut (1964).

Let

$$\begin{array}{ll}
\alpha \equiv & A\ B\ C\ D\ \ldots\ U\ V \\
\beta \equiv & V\ A\ B\ C\ \ldots\ T\ U \\
\gamma \equiv & U\ V\ A\ B\ \ldots\ S\ T \\
\vdots & \vdots \\
\psi \equiv & D\ E\ F\ G\ \ldots\ B\ C \\
\phi \equiv & C\ D\ E\ F\ \ldots\ A\ B \\
\omega \equiv & B\ C\ D\ E\ \ldots\ V\ A
\end{array}$$

where A, B, ..., V are the t treatments.

If t is odd, the design for t treatments can be written symbolically as in Table 4.2. In Table 4.2, each column represents t subjects.

Table 4.2 SF design by Berenblut (1964) for t treatments.

		Subject (1 to t^2)			
	1	α	α	...	α
	2	β	γ	...	α
	3	γ	γ	...	γ
	4	δ	ϵ	...	γ
	5	ϵ	ϵ	...	ϵ
	$\vdots$	$\vdots$	$\vdots$	...	$\vdots$
Period	t-1	ϕ	ω	...	ψ
	t	ω	ω	...	ω
	t+1	ω	α	...	ϕ
	t+2	ϕ	ϕ	...	ϕ
	$\vdots$	$\vdots$	$\vdots$	...	$\vdots$
	2t-1	β	β	...	β
	2t	α	β	...	ω

If t is even, the lines for periods t and $t+1$, for periods $t-1$ and $t+2$ etc., are interchanged.

The following variations of the design are possible without upsetting its orthogonality properties.

(i) Rows with odd numbers may be permuted amongst themselves ($t!$ permutations)

(ii) Rows with even numbers may be permuted amongst themselves ($t!$ permutations)

(iii) The design may be read in reverse order of time.

4.3 Method of construction of designs by Patterson (1970).

In this section, we briefly describe the construction of designs by Patterson (1970). This is useful for an understanding of the construction of a wider class of designs in the later sections of this chapter.

The method for $t=4$ is given below. This can obviously be extended to any number of treatments. It consists of

(1) writing down the 16 combinations of levels in period 1 and period 2

(2) permuting the levels in period 1 and period 2 to obtain the levels for the remaining odd-numbered periods and the even-numbered periods respectively, the permutations being provided by two Latin squares of numbers 1, 2, 3, 4 with the first row in each square in standard order. These squares are called generating squares and denoted by G and H. The treatments of the remaining odd-numbered periods i in the complete design are obtained by rearranging the treatments of period 1 in the order of row $^{(i+1)}/_2$ of G; the treatments of even-numbered periods i are obtained by rearranging the treatments of period 2 in the order of row $^{i}/_2$ of H.

For example, the design in Table 4.1 is generated by the Latin squares shown in Table 4.3 .

Table 4.3 Latin squares generating the design of Table 4.1

	Square G					Square H			
Period	Treatments				Period	Treatments			
1	1	2	3	4	2	1	2	3	4
3	4	3	2	1	4	3	4	1	2
5	2	1	4	3	6	2	1	4	3
7	3	4	1	2	8	4	3	2	1

It follows from the method of construction that the design includes all combinations of treatments in any two consecutive periods. Therefore, the design has serial factorial property.

Since any Latin square can be obtained by permuting the rows and columns of a subset of the transformation sets, the new designs can be classified into design classes. There are two transformation sets of 4×4 Latin squares, numbered I and II . The three subsets in transformation set I are labelled A, B, C and the single subset in transformation set II is labelled D . Any square of a subset can be derived from any other square of the subset by permutation of rows and columns. Each subset has exactly one standard square with first row and first column both in standard order 1 2 3 4 . The standard squares are given in Table 4.4. (See Fisher and Yates (1963, Table XV)).

Table 4.4 Standard 4×4 Latin squares.

Transformation set I			Transformation set II
	Subset		Subset
A	B	C	D
1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4
2 1 4 3	2 3 4 1	2 4 1 3	2 1 4 3
3 4 2 1	3 4 1 2	3 1 4 2	3 4 1 2
4 3 1 2	4 1 2 3	4 3 2 1	4 3 2 1

A generating square can be represented by the subset label and first column. Thus the squares G and H of Table 4.3 are D(1423) and D(1324).

There are 16 classes of designs. Designs of class II $(I = A, B, C, D)$ are simple designs with both generating squares selected from subset I. Designs of class I J are mixed designs with square G selected from subset I and square H from a different subset J.

A complete design can be concisely represented by the design class, the first column of G and the first column of H. For example, the design of Table 4.1 is DD(1423)(1324).

Designs for four treatments obtained by Berenblut (1964) belong to design classes AA, BB and CC. Designs of any two of these classes can be obtained by permuting the treatment levels of designs in the third class. Designs of class DD and mixed designs are not obtained by Berenblut's method.

For $t = 2$ and $t = 3$, the designs generated by Patterson (1970) are identical with Berenblut's designs (1964). But for $t > 3$, Patterson's method always gives additional designs. We have already seen that the design with four treatments in Table 4.1 cannot be obtained by Berenblut's method.

4.4 General SF designs.

Consider the SF designs for t treatments with the t^2 subjects divided into t groups of t subjects each. Let the t treatments in the i^{th} period and j^{th} group be represented by a $t \times t$ matrix M_{ij} with cell $(l, m) = 1$ if subject m receives treatment l and cell $(l, m) = 0$ otherwise. Then the most general SF design can be written with such matrices as in Table 4.5.

Table 4.5 Matrix representation of the general SF design.

	Group						
Period	1	2	...	j	...	t
1	M_{11}	M_{12}	...	M_{1j}	...	M_{1t}
2	M_{21}	M_{22}		M_{2j}		M_{2t}
⋮	⋮	⋮				⋮
i	M_{i1}	M_{i2}		M_{ij}		M_{it}
⋮	⋮	⋮				⋮
2t	$M_{2t,1}$	$M_{2t,2}$	...	$M_{2t,j}$	...	$M_{2t,t}$

In Table 4.5, the matrices M_{ij}'s are related to the first-order incidence matrices and therefore second-order incidence matrices. We have

$$X_i^T = (M_{i1}\ M_{i2}\ \dots\ M_{it})\ ,\ i = 1, 2, \dots, 2t,$$

so that $X_i^T X_i = \left(\sum_{j=1}^{t} M_{ij} M_{ij}^T \right)$.

Since $\quad X_i = X_{i,\, i-1}\, Q_1$ (see subsection 3.2.4),

therefore

$$Q_1^T X_{i,\, i-1}^T = (M_{i1}\ M_{i2}\ \dots\ M_{it})\ ,\quad i = 1, 2, \dots, 2t\ ,$$

where $\quad X_{10} = \frac{1}{t} (X_1 \otimes 1_{(t)}^T)$, The combinatorial conditions of SF designs (see subsection 3.4.1) require

$$1. \quad \sum_{j=1}^{t} M_{i-1,j} M_{ij}^T = J_{(t)} \;, \quad i = 2, 3, \ldots, 2t \;, \qquad (4.1)$$

$$2. \quad \sum_{i=1}^{2t} M_{ij} = 2 J_{(t)} \;, \quad j = 1, 2, \ldots, t \;. \qquad (4.2)$$

Condition (4.1) ensures that each combination of treatment levels in periods $i-1, i$ occurs on one subject $(i = 2, 3, \ldots, 2t)$. Condition (4.2) ensures that each treatment level occurs twice with each subject.

Three particular systems of the general SF designs will be distinguished.

4.5 Three systems of the general SF designs.

In this section, three systems of the general G-class designs will be considered. In these, the M_{ij} matrices take particular forms.

4.5.1 System A of SF designs.

This system of the general SF designs can be written as in Table 4.6 below.

Table 4.6 System A of SF designs.

	group						
Period	1	2	$\cdots$	j	$\cdots$	t
1	$I_{(t)}$	$I_{(t)}$	$\cdots$	$I_{(t)}$	$\cdots$	$I_{(t)}$
2	D_{11}	D_{12}		D_{1j}		D_{1t}
3	P_{21}	P_{22}		P_{2j}		P_{2t}
4	D_{21}	D_{22}		D_{2j}		D_{2t}
$\vdots$	$\vdots$	$\vdots$				$\vdots$
i_1 (odd)	$P_{i_1',1}$	$P_{i_1',2}$		$P_{i_1',j}$		$P_{i_1',t}$
i_2 (even)	$D_{i_2',1}$	$D_{i_2',2}$		$D_{i_1',j}$		$D_{i_2',t}$
$\vdots$	$\vdots$	$\vdots$				$\vdots$
2t-1	P_{t1}	P_{t2}		P_{tj}		P_{tt}
2t	D_{t1}	D_{t2}	$\cdots$	D_{tj}	$\cdots$	D_{tt}

In Table 4.6, $i_1' = (i_1 + 1)/2$,

$i_2' = {}^{i_2}/_2$.

For simplicity, in this subsection and the next, let the general period of the odd-numbered periods be period $i(i = 1, 2, \ldots, t)$ and the general period of the even-numbered periods be also period $i(i = 1, 2, \ldots, t)$.

For System A designs, the matrices M_{ij} take a special form, P_{ij}, in odd-numbered periods and another special form, D_{ij}, in even-numbered periods. P_{ij} is a permutation matrix for $i, j = 1, 2, \ldots, t$. In particular, we put $P_{1j} = I_{(t)}$ for $j = 1, 2, \ldots, t$. $D_{ij}\ (i, j = 1, 2, \ldots, t)$. is a $t \times t$ matrix, one row of which is a vector of 1's ; the other rows are zero vectors. Such a matrix is said to be of <u>type S</u> if the S^{th} row is the vector of 1's .

In Table 4.6, there is a special standardised order of subjects. If D_{11} is of type 1 , D_{12} is of type 2, ..., D_{1t} is of type t , then the subjects are arranged as in Order B (see subsection 3.4.4 of Chapter Three).

There is some relationship between the presentation of System A designs in Table 4.6 and the presentation of Berenblut's designs (1964) in Table 4.2 . In Table 4.2 , each symbol represents the treatments for t subjects in a period. But in Table 4.6 , the treatments for t subjects in a period are represented by a matrix, P_{ij} or D_{ij} . This matrix representation is more general than the representation by symbols. We will show later that designs by Berenblut (1964) and even designs by Patterson (1970) belong to System A .

In System A designs, P_{ij} and D_{ij} are such that

$$\sum_{i=1}^{t} P_{ij} = J_{(t)} \quad , \qquad j = 1, 2, \ldots, t \qquad\qquad (4.3)$$

$$\sum_{i=1}^{t} D_{ij} = J_{(t)} \quad , \quad j = 1, 2, \ldots, t \qquad (4.4)$$

$$\sum_{j=1}^{t} D_{ij} = J_{(t)} \quad , \quad i = 1, 2, \ldots, t \qquad (4.5)$$

The combinatorial conditions of SF designs require

$$1. \quad \sum_{j=1}^{t} P_{ij} D_{ij}^{T} = J_{(t)} \quad , \quad i = 1, 2, \ldots, t \qquad (4.6a)$$

$$\text{and} \quad \sum_{j=1}^{t} D_{i-1,j} P_{ij}^{T} = J_{(t)} , \quad i = 2, 3, \ldots, t \qquad (4.6b)$$

$$2. \quad \sum_{i=1}^{t} (P_{ij} + D_{ij}) = 2J_{(t)} , \; j = 1, 2, \ldots, t \qquad (4.7)$$

These are satisfied by System A designs through (4.3), (4.4) and (4.5).

Condition (4.3) can be interpreted as follows. The treatments in odd-numbered periods of each group constitute a Latin square. The same or different Latin squares may be used for different groups. Thus a maximum of t Latin squares may be used in odd-numbered periods for a design with t treatments. Conditions (4.4) and (4.5) mean that the treatments in even-numbered periods constitute a single Latin square repeated t times.

Condition (4.5) requires that one of the matrices D_{i1}, D_{i2}, ..., D_{it} is of type 1, one is of type 2, ..., and one is of type t. When $i = 1$, (4.5) ensures that all combinations occur in periods 1 and 2.

A design of System A that is not in Systems B or C (see the two following subsections 4.5.2 and 4.5.3) is given in Table 4.13. (The notation in the table will be explained in subsection 4.6.2). Two Latin squares are used for odd-numbered periods. They are

$$\begin{matrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 3 & 4 & 1 & 2 \\ 2 & 1 & 4 & 3 \end{matrix} \qquad \begin{matrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \\ 2 & 1 & 4 & 3 \\ 4 & 3 & 2 & 1 \end{matrix}$$

The single Latin square used for even-numbered periods is the second Latin square given above. (Note that this is not essential). We could have used three or four Latin squares for odd-numbered periods but can only use one Latin square for even-numbered periods.

All designs constructed by Berenblut (1964) and Patterson (1970) for $t \geq 2$ belong to System A. For Berenblut's designs, the treatments in odd-numbered periods of every group constitute the same $t \times t$ Latin square from any subset of only one fixed transformation set of $t \times t$ Latin squares. (It is transformation set I for $t=4$ (see Table 4.4)). The treatments in even-numbered periods constitute any $t \times t$ Latin square, repeated t times, in the same subset of the same transformation set.

In the case of Patterson's designs, the treatments in odd-numbered periods of every group constitute the same $t \times t$ Latin square from a subset of any transformation set. (They are transformation sets I and II for $t=4$ (see Table 4.4)). The treatments in even-numbered periods constitute any $t \times t$ Latin square, repeated t times, from a subset of any transformation set. Therefore, the design in Table 4.13 does not belong to the class of designs by Berenblut (1964) and Patterson (1970) for four treatments.

We now show in Theorem 4.1 below that all System A designs are R-orthogonal.

<u>Theorem 4.1</u> All SF designs in System A are R-orthogonal.

Proof:

The result follows immediately by application of Theorem 3.5 of Chapter Three. Condition (1) of Theorem 3.5 is satisfied because

$$\sum_{j=1}^{t} I_{(t)} D_{tj}^{T} = J_{(t)} \quad .$$

It can be checked that the required frequencies in the interpretation of the second condition, $T_2NN^T = T_2$, of Theorem 3.5 (see subsection 3.6.2) are satisfied by all SF designs in System A.

Corollary: 1. Designs constructed by Berenblut (1964) are R-orthogonal.

Corollary: 2. Designs constructed by Patterson (1970) are also R-orthogonal.

The construction of designs of this system for three and four treatments is given in section 4.6.

4.5.2 System B of SF designs.

This system of the general SF designs can be written as in Table 4.7.

Table 4.7 System B of SF designs.

	group						
Period	1	2	...	j	...	t
1	D_{11}	D_{12}	...	D_{1j}	...	D_{1t}
2	$I_{(t)}$	$I_{(t)}$		$I_{(t)}$		$I_{(t)}$
3	D_{21}	D_{22}		D_{2j}		D_{2t}
4	P_{21}	P_{22}		P_{2j}		P_{2t}
$\vdots$	$\vdots$	$\vdots$				$\vdots$
i_1 (odd)	$D_{i_1',1}$	$D_{i_1',2}$		$D_{i_1',j}$		$D_{i_1',t}$
i_2 (even)	$P_{i_2',1}$	$P_{i_2',2}$		$P_{i_2',j}$		$P_{i_2',t}$
$\vdots$	$\vdots$	$\vdots$				$\vdots$
2t-1	D_{t1}	D_{t2}		D_{tj}		D_{tt}
2t	P_{t1}	P_{t2}	...	P_{tj}	...	P_{tt}

In Table 4.7, $i_1' = (i_1 + 1)/2$, $i_2' = i_2/2$. There is also a special standardised order of subjects. If D_{11} is of type 1, D_{12} is of type 2, ..., D_{1t} is of type t, then the subjects are arranged as in Order A (see subsection 3.4.4 of Chapter Three). P_{ij}'s are permutation matrices. We put $P_{1j} = I_{(t)}$, $j = 1, 2, \ldots, t$. D_{ij}'s are as defined for System A. In System B designs, P_{ij} and D_{ij} are such that they satisfy (4.3), (4.4) and (4.5) in subsection 4.5.1.

The combinatorial conditions of SF designs require

$$\sum_{j=1}^{t} D_{ij} P_{ij}^{T} = J_{(t)}, \quad i = 1, 2, \ldots, t, \tag{4.8a}$$

$$\sum_{j=1}^{t} P_{i-1,j} D_{ij}^{T} = J_{(t)}, \quad i = 2, 3, \ldots, t, \tag{4.8b}$$

and
$$\sum_{i=1}^{t} (D_{ij} + P_{ij}) = 2 J_{(t)}, \quad j = 1, 2, \ldots, t, \tag{4.9}$$

These are satisfied by System B designs through (4.3), (4.4) and (4.5).

Unlike System A designs, Conditions (4.3), (4.4) and (4.5) are interpreted differently here. Condition (4.3) means that the treatments in even-numbered periods (as opposed to odd-numbered periods in System A) of each group constitute a Latin square with the same or different Latin squares used for different groups. Conditions (4.4) and (4.5) mean that the treatments in odd-numbered periods (as opposed to even-numbered periods in System A) constitute a single Latin square, repeated t times. Condition (4.5) is interpreted in the same way as for System A but for odd-numbered periods. Therefore, the design of System A in Table 4.13 is not in System B.

Not all designs of this system are R-orthogonal. A design that is not R-ortho is given in Table 4.17. (The notation in the table will be explained in subsection 4.7.2). As in the case of designs of System A, it can be shown that designs by Berenblut (1964) and Patterson (1970) belong to System B. Thus there is some overlapping of designs of this system and those of System A. These 'overlapped' designs are obviously R-orthogonal. However, not all R-ortho designs of System B are also in System A. An R-ortho design that is not an 'overlapped' design with System A or even with System C (see the immediately following subsection 4.5.3) is given in Table 4.16. (Again, the notation in the table will be explained in subsection 4.7.2). This design is not in System A because more than one Latin square is required for even-numbered periods.

4.5.3 System C of SF designs.

This system of the general SF designs can be written as in Table 4.8.

Table 4.8 System C of SF designs.

	group						
Period	1	2	...	j	...	t
1	$I_{(t)}$	$I_{(t)}$	...	$I_{(t)}$	...	$I_{(t)}$
2	P_{21}	P_{22}		P_{2j}		P_{2t}
$\vdots$	$\vdots$					$\vdots$
i	P_{i1}	P_{i2}		P_{ij}		P_{it}
$\vdots$	$\vdots$					$\vdots$
2t	$P_{2t,1}$	$P_{2t,2}$	...	$P_{2t,j}$	...	$P_{2t,t}$

In Table 4.8, P_{ij}'s are $t \times t$ permutation matrices such that

$$\sum_{\text{over } i \text{ (odd)}} P_{ij} = J_{(t)} \quad , \quad j = 1, 2, \dots, t \, , \qquad (4.10)$$

$$\sum_{\text{over } i \text{ (even)}} P_{ij} = J_{(t)} \quad , \quad j = 1, 2, \dots, t \, , \qquad (4.11)$$

$$\sum_{j=1}^{t} P_{i-1,j} P_{ij}^{T} = J_{(t)} \quad , \quad i = 2, 3, \dots, 2t \, . \qquad (4.12)$$

We put $P_{1j} = I_{(t)}$ for $j = 1, 2, \dots, t$. The combinatorial conditions of SF designs require that (4.12) holds and

$$\sum_{i=1}^{2t} P_{ij} = 2\, J_{(t)} \quad , \quad j = 1, 2, \ldots, t \ . \qquad (4.13)$$

(4.13) follows immediately from (4.10) and (4.11) .

As with System B , not all designs of this system are R-orthogonal. A design that is not R-orthogonal is given in Table 4.21. (The notation in the table will be explained in subsection 4.8.2). As with designs of Systems A and B , it can be shown that designs by Berenblut (1964) and Patterson (1970) also belong to this system. Thus designs by Berenblut (1964) and Patterson (1970) belong to all three systems of designs. Therefore, there is some overlapping of designs of this system and those of Systems A and B . Those 'overlapped' designs with System A are obviously R-orthogonal while those 'overlapped' designs with System B are not necessarily R-orthogonal. Not all R-ortho designs of this system are 'overlapped' designs with System A or System B . An R-ortho design that is not in either System A or System B is given in Table 4.20 . (Again, the notation in the table is explained in subsection 4.8.2) . In this design, more than one Latin square is required to generate the treatments of odd-numbered periods and more than one Latin square is also required for even-numbered periods.

From the representation of System C designs in Table 4.8 , it is readily seen that the t^2 subjects of any design of System C can be arranged in t groups of t subjects each such that every treatment occurs within any period of a group. Not all designs of System A or System B can have their subjects arranged in this way. Indeed it is not at all obvious which design of these two systems can be so arranged. A design of System A that cannot be so arranged is in Table 4.13. Therefore, this design is not in System C. Similarly, it can be shown that the design of System B in Table 4.16 is also not in

System C . Hence all System C designs have an advantage in that the t^2 subjects of any of these designs can be arranged in blocks of t subjects without confounding direct or residual effects (ignoring interaction) with blocks.

4.6 Construction of designs of System A.

In this section, we construct designs of System A for three and four treatments.

4.6.1 Designs for three treatments.

There are sixteen designs for three treatments in System A . These include the four designs constructed by Berenblut (1964). Patterson's method gives the same four designs as Berenblut's method. Therefore there are twelve new designs. One of these is in Table 4.9 .

Table 4.9 A new System A design for three treatments.

Design $A_1(132)R_1(123)A_2(132)R_2(231)A_3(123)R_3(312)$.

	Group								
	1			2			3		
	Subject			Subject			Subject		
Period	I	II	III	I	II	III	I	II	III
1	1	2	3	1	2	3	1	2	3
2	1	1	1	2	2	2	3	3	3
3	3	1	2	3	1	2	2	3	1
4	2	2	2	3	3	3	1	1	1
5	2	3	1	2	3	1	3	1	2
6	3	3	3	1	1	1	2	2	2

There is only one standard 3×3 Latin square. This is given in Table 4.10 .

Table 4.10 Standard 3×3 Latin square.

1	2	3
2	3	1
3	1	2

The design in Table 4.9 can be represented by $A_1(132)\ R_1(123)\ A_2(132)\ R_2(231)\ A_3(123)\ R_3(312)$ where $A_1(132)$ represents the generating square used for the odd-numbered periods of the first group and $R_1(123)$ means that the treatments of all three subjects of the first group are 1 in period 2, 2 in period 4 and 3 in period 6. The remaining terms in the representation are similarly explained. Therefore, the single Latin square used to generate treatments in even-numbered periods is the standard 3×3 Latin square in Table 4.10. The representations for all 16 designs are given in Table 4.11.

Table 4.11 Representations of all possible System A designs for three treatments.

Berenblut's designs

1. $A_1(123)\ R_1(123)\ A_2(123)\ R_2(231)\ A_3(123)\ R_3(312)$
2. $A_1(132)\ R_1(123)\ A_2(132)\ R_2(231)\ A_3(132)\ R_3(312)$
3. $A_1(123)\ R_1(132)\ A_2(123)\ R_2(213)\ A_3(123)\ R_3(321)$
4. $A_1(132)\ R_1(132)\ A_2(132)\ R_2(213)\ A_3(132)\ R_3(321)$

New designs

1. $A_1(132)\ R_1(123)\ A_2(132)\ R_2(231)\ A_3(123)\ R_3(312)$
2. $A_1(123)\ R_1(132)\ A_2(132)\ R_2(213)\ A_3(132)\ R_3(321)$
3. $A_1(123)\ R_1(123)\ A_2(123)\ R_2(231)\ A_3(132)\ R_3(312)$
4. $A_1(132)\ R_1(132)\ A_2(123)\ R_2(213)\ A_3(123)\ R_3(321)$
5. $A_1(132)\ R_1(123)\ A_2(123)\ R_2(231)\ A_3(132)\ R_3(312)$

Table 4.11 continued

6. $A_1(132)$ $R_1(132)$ $A_2(132)$ $R_2(213)$ $A_3(123)$ $R_3(321)$

7. $A_1(123)$ $R_1(123)$ $A_2(132)$ $R_2(231)$ $A_3(123)$ $R_3(312)$

8. $A_1(123)$ $R_1(132)$ $A_2(123)$ $R_2(213)$ $A_3(132)$ $R_3(321)$

9. $A_1(123)$ $R_1(123)$ $A_2(132)$ $R_2(231)$ $A_3(132)$ $R_3(312)$

10. $A_1(132)$ $R_1(132)$ $A_2(123)$ $R_2(213)$ $A_3(132)$ $R_3(321)$

11. $A_1(132)$ $R_1(123)$ $A_2(123)$ $R_2(231)$ $A_3(123)$ $R_3(312)$

12. $A_1(123)$ $R_1(132)$ $A_2(132)$ $R_2(213)$ $A_3(123)$ $R_3(321)$

4.6.2 Designs for four treatments.

We first define First design, basic design classes (basic designs) and complex design.

Def$^{\underline{n}}$ 4.1 : A design constructed by Patterson's method in section 4.3 with generating squares G and H both standard squares will be called a First design. Each design class has just one First design. (Generating squares, standard squares and design classes are defined in section 4.3). For example, the First design of design class DD for four treatments is DD(1234)(1234).

Def$^{\underline{n}}$ 4.2 : The design classes of Patterson (1970)(see section 4.3) will be known as basic design classes. Any particular design of the basic design classes will be known as a basic design. That is, a basic design is obtained when one Latin square is used to generate the treatments in the odd-numbered periods and also only one Latin square for even-numbered periods.

The above two definitions are general for any number of treatments. In particular, the sixteen simple and mixed design classes IJ (I, J = A, B, C, D) constructed by Patterson (1970) for four treatments are known as basic design classes.

Def$^{\underline{n}}$ 4.3 : A design is called a <u>complex design</u> if more than one Latin square is used to generate the treatments in the odd-numbered periods or more than one Latin square is used for the even-numbered periods or both. (A complex design class will be defined later). A complex design, therefore, does not belong to the class of designs constructed by Berenblut (1964) and Patterson (1970). For example, the design in Table 4.13 is a complex design. It is shown in subsection 4.5.1 that two Latin squares are used for the odd-numbered periods of this design.

This definition is general for any number of treatments. However, in this subsection, we are only concerned with the construction of complex designs for four treatments. Complex designs can be obtained in three ways by partitioning the 16 subjects of the basic designs. These three partitions do not exhaust all the complex designs of System A for four treatments.

<u>Partition A1</u>.

Partition A1 involves the partitioning of the 16 subjects of any basic design into 2 groups: Group I consists of subjects with sequences 11, 21, 31, 41, 14, 24, 34, 44 in periods 1 and 2. Group II contains the other 8 subjects.

Twelve basic design classes are divided into 3 sets as follows:

Set 1 : DD, CC, BD, AD

Set 2 : BB, DB, CB, AB

Set 3 : AA, DA, CA, BA

We now consider the following combination.

<u>Combination W</u> : Group I of the First design of a basic design class in any set is combined with Group II of any of the other First designs in the same set.

Complex designs are obtained by performing Combination W on the above 3 sets. For example, if Group I of the First design of basic design class DD in Set 1 is combined with Group II of the First design of basic design class CC, a complex design is obtained. This is given in Table 4.12. (The notation in the table will be explained later in this subsection). This complex design required two Latin squares to generate the treatments in the odd-numbered periods. These Latin squares are the standard squares of Subsets C and D (see Table 4.4). The single Latin square for even-numbered periods is the standard square of Subset C.

Table 4.12 Design A1: $(DD)_1(1234)(1234)(CC)_2(1234)(1234)$.

	Subject															
Period	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII	XIV	XV	XVI
1	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
2	1	1	1	1	4	4	4	4	2	2	2	2	3	3	3	3
3	2	1	4	3	2	1	4	3	2	4	1	3	2	4	1	3
4	2	2	2	2	3	3	3	3	4	4	4	4	1	1	1	1
5	3	4	1	2	3	4	1	2	3	1	4	2	3	1	4	2
6	3	3	3	3	2	2	2	2	1	1	1	1	4	4	4	4
7	4	3	2	1	4	3	2	1	4	3	2	1	4	3	2	1
8	4	4	4	4	1	1	1	1	3	3	3	3	2	2	2	2

Each complex design obtained above will be known as the First design of a complex design class. The other designs of the complex design class are obtained from the First design by the following permutation.

Permutation P : a) permute the odd-numbered periods 3, 5 and 7 of all 16 subjects.

b) Permute the even-numbered periods 4, 6 and 8 of all 16 subjects.

A complex design class, therefore, has just one First design. As in basic designs, the 16 subjects of a complex design can be partitioned into 2 groups according to Partition A1 . For example, Subjects I - VIII of the complex design in Table 4.12 constitute Group I : other subjects constitute Group II .

We now consider the following two operations.

Operation QA : For the First design of each of the basic design classes in Set 1 , and for each of the First designs of complex design classes obtained by performing Combination W on Set 1 , permute odd-numbered periods 3, 5, and 7, and permute two of the even-numbered periods 4, 6 and 8 of its Group II .

Operation RA : For the First design of each of the basic design classes in Sets 2 and 3 , and for each of the First designs of complex design classes obtained by performing Combination W on Set 2 and Set 3 , permute odd-numbered periods 3, 5 and 7 of its Group II .

Additional complex designs can be obtained by performing Operation QA (the 2 even-numbered periods being periods 4 and 6) and Operation RA . Each of these complex designs will also be known as the First design of a complex design class. The other designs of the complex design class are obtained from the First design by performing Permutation P . As an example, eleven additional First designs of complex design classes can be obtained by permuting periods 3, 5 and 7 and permuting periods 4 and 6 of Group II of the design in Table 4.12 .

Representation of designs.

The method of representation of First designs of complex design classes will be explained by means of an example. The design in

Table 4.12 obtained by combining Group I of the First design of basic design class DD and Group II of the First design of basic design class CC will be represented by A1 : $(DD)_1(1234)(1234)$ $(CC)_2(1234)(1234)$. In the representation, A1 represents the partition used (A1 in this case); $(DD)_1(1234)(1234)(CC)_2(1234)(1234)$ means that the design consists of Group I of basic design DD(1234)(1234) and Group II of basic design CC(1234)(1234). Such a representation completely determines the design. The eleven First designs of complex design classes obtained by permuting periods 3, 5 and 7 and permuting periods 4 and 6 of Group II of the design in Table 4.12 are therefore represented by

1. A1 : $(DD)_1(1234)(1234)(CC)_2(1324)(1234)$
2. A1 : $(DD)_1(1234)(1234)(CC)_2(1342)(1234)$
3. A1 : $(DD)_1(1234)(1234)(CC)_2(1432)(1234)$
4. A1 : $(DD)_1(1234)(1234)(CC)_2(1423)(1234)$
5. A1 : $(DD)_1(1234)(1234)(CC)_2(1243)(1234)$
6. A1 : $(DD)_1(1234)(1234)(CC)_2(1234)(1324)$
7. A1 : $(DD)_1(1234)(1234)(CC)_2(1324)(1324)$
8. A1 : $(DD)_1(1234)(1234)(CC)_2(1342)(1324)$
9. A1 : $(DD)_1(1234)(1234)(CC)_2(1432)(1324)$
10. A1 : $(DD)_1(1234)(1234)(CC)_2(1423)(1324)$
11. A1 : $(DD)_1(1234)(1234)(CC)_2(1243)(1324)$.

For example, design 8. consists of Group I of basic design (DD)(1234)(1234) and Group II of basic design CC(1342)(1324) . Hence, all the First designs of complex design classes obtained can be represented in this way.

The same representation can be used for individual complex designs but with appropriate first column of generating square G and first column of generating square H for both basic designs used. For example, the design in Table 4.13 is A1 : $(DD)_1(1432)(1324)$ $(DD)_2(1324)(1324)$. That is, it is obtained by combining Group I of basic design DD(1432)(1324) and Group II of basic design DD(1324)(1324). Subjects I - VIII in Table 4.13 constitute Group I; other subjects constitute Group II. These two groups could be interchanged. In other words, A1 : $(DD)_2(1324)(1324)(DD)_1(1432)(1324)$ gives exactly the same design. However, for simplicity, we will adhere to the former representation throughout the rest of the thesis.

A complex design class will be represented by its First design. Therefore, the First design A1 : $(DD)_1(1234)(1234)(CC)_2(1234)(1234)$ and other designs obtained from it by performing Permutation P belong to complex design class A1 : $(DD)_1(1234)(1234)(CC)_2(1234)(1234)$.

Table 4.13 Design A1 : $(DD)_1(1432)(1324)(DD)_2(1324)(1324)$ for four treatments.

	Subject															
Period	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII	XIV	XV	XVI
1	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
2	1	1	1	1	4	4	4	4	2	2	2	2	3	3	3	3
3	4	3	2	1	4	3	2	1	3	4	1	2	3	4	1	2
4	3	3	3	3	2	2	2	2	4	4	4	4	1	1	1	1
5	3	4	1	2	3	4	1	2	2	1	4	3	2	1	4	3
6	2	2	2	2	3	3	3	3	1	1	1	1	4	4	4	4
7	2	1	4	3	2	1	4	3	4	3	2	1	4	3	2	1
8	4	4	4	4	1	1	1	1	3	3	3	3	2	2	2	2

Partition A2.

Partition A2 involves the partitioning of the 16 subjects of any basic design into 2 groups. Since the 2 groups are associated with this partition, we will still call them Group I and Group II.

Group I contains subjects with sequences 11, 21, 31, 41, 13, 23, 33 and 43 in periods 1 and 2 : the other 8 subjects constitute Group II.

Twelve basic design classes are divided into 3 sets as follows:

Set 1 : DD, BB, CD, AD

Set 2 : CC, DC, BC, AC

Set 3 : AA, DA, CA, BA .

As in Partition A1 , First designs of complex design classes are obtained by performing Combination W on these 3 sets. Additional First designs of complex design classes are obtained by performing Operation QA (the 2 even-numbered periods are periods 4 and 8) and Operation RA .

Partition A3.

In Partition A3, Group I of any basic design contains subjects with sequences 11, 21, 31, 41, 12, 22, 32, 42 in periods 1 and 2 . Group II contains the other 8 subjects.

Twelve basic design classes are divided into 3 sets as follows:

Set 1 : DD, AA, CD, BD

Set 2 : CC, DC, BC, AC

Set 3 : BB, DB, CB, AB .

First designs of complex design classes are obtained by performing Combination W on these 3 sets. Additional First designs of complex design classes are obtained by performing Operation QA (the 2 even-numbered periods are periods 6 and 8) and Operation RA .

4.7 Construction of designs of System B.

In this section, we construct designs of System B for three and four treatments.

4.7.1 Designs for three treatments.

System B has sixteen designs for three treatments. These designs include the four designs constructed by Berenblut (1964). Therefore, there are twelve new designs, one of which is in Table 4.14.

Table 4.14 Design $R_1(123)\ B_1(132)\ R_2(231)\ B_2(132)\ R_3(312)\ B_3(123)$ for three treatments of System B.

	Group								
	1			2			3		
	Subject			Subject			Subject		
Period	I	II	III	I	II	III	I	II	III
1	1	1	1	2	2	2	3	3	3
2	1	2	3	1	2	3	1	2	3
3	2	2	2	3	3	3	1	1	1
4	3	1	2	3	1	2	2	3	1
5	3	3	3	1	1	1	2	2	2
6	2	3	1	2	3	1	3	1	2

The design in Table 4.14 can be represented as $R_1(123)\ B_1(132)\ R_2(231)\ B_2(132)\ R_3(312)\ B_3(123)$ where $R_1(123)$ means that the treatments of all three subjects in Group I are 1 in period 1, 2 in period 3 and 3 in period 5 while $B_1(132)$ refers to the generating square used for the even-numbered periods of Group I. The other four terms in the representation are explained in the similar manner. Therefore, the single Latin square used to generate the treatments in the odd-numbered periods is the standard 3×3 Latin square in Table 4.10. The representations for the twelve new designs are given in Table 4.15. All these designs are not R-orthogonal and are therefore not in System A. Thus, Berenblut's designs (1964) are the only designs for three treatments that are common to System A and System B.

Table 4.15 Representations of the 12 new System B designs for 3 treatments.

Design	
1.	$R_1(123)\ B_1(132)\ R_2(231)\ B_2(132)\ R_3(312)\ B_3(123)$
2.	$R_1(132)\ B_1(132)\ R_2(213)\ B_2(132)\ R_3(321)\ B_3(123)$
3.	$R_1(123)\ B_1(123)\ R_2(231)\ B_2(123)\ R_3(312)\ B_3(132)$
4.	$R_1(132)\ B_1(123)\ R_2(213)\ B_2(123)\ R_3(321)\ B_3(132)$
5.	$R_1(123)\ B_1(132)\ R_2(231)\ B_2(123)\ R_3(312)\ B_3(132)$
6.	$R_1(132)\ B_1(132)\ R_2(213)\ B_2(123)\ R_3(321)\ B_3(132)$
7.	$R_1(123)\ B_1(123)\ R_2(231)\ B_2(132)\ R_3(312)\ B_3(123)$
8.	$R_1(132)\ B_1(123)\ R_2(213)\ B_2(132)\ R_3(321)\ B_3(123)$
9.	$R_1(123)\ B_1(123)\ R_2(231)\ B_2(132)\ R_3(312)\ B_3(132)$
10.	$R_1(132)\ B_1(123)\ R_2(213)\ B_2(132)\ R_3(321)\ B_3(132)$
11.	$R_1(123)\ B_1(132)\ R_2(231)\ B_2(123)\ R_3(312)\ B_3(123)$
12.	$R_1(132)\ B_1(132)\ R_2(213)\ B_2(123)\ R_3(321)\ B_3(123)$

4.7.2 Designs for four treatments.

Complex designs can be obtained in three ways by partitioning the 16 subjects of the basic designs. Note that these three partitions do not exhaust all the designs of System B.

Partition B1.

In this partition, Group I of any basic design contains subjects with sequences 11, 12, 13, 14, 41, 42, 43, 44 in periods 1 and 2 : the other 8 subjects constitute Group II.

Twelve basic design classes are divided into 3 sets as follows:

Set 1 : DD, CC, DB, DA

Set 2 : BB, BD, BC, BA

Set 3 : AA, AD, AC, AB .

As in Partitions A1, A2 and A3, First designs of complex design classes are obtained by performing Combination W (defined in subsection 4.6.2) on these 3 sets.

We now consider the following two operations.

Operation QB : For the First design of each of the basic design classes in Set 1, and for each of the First designs of complex design classes obtained by performing Combination W (defined in subsection 4.6.2) on Set 1, permute even-numbered periods 4, 6 and 8, and permute two of the odd-numbered periods 3, 5 and 7 of its Group II.

Operation RB : For the First design of each of the basic design classes in Sets 2 and 3, and for each of the First designs of complex design classes obtained by performing Combination W (defined in subsection 4.6.2) on Set 2 and Set 3, permute even-numbered periods 4, 6 and 8 of its Group II.

Additional complex designs can be obtained by performing Operation QB (the 2 odd-numbered periods being periods 3 and 5) and Operation RB. Each of these complex designs will again be known as the First design of a complex design class. The other designs of the complex design class are obtained from the First design by performing Permutation P (defined in subsection 4.6.2).

Designs constructed by Partition B1 are represented in the same way as those by Partitions A1, A2 and A3. An R-ortho design constructed by Partition B1 is given in Table 4.16. Table 4.17 contains a design that is not R-orthogonal.

Table 4.16 R-ortho design B1 : $(CC)_1(1342)(1324)(CC)_2(1243)(1432)$ for four treatments.

Subject

Period	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII	XIV	XV	XVI
1	1	1	1	1	4	4	4	4	2	2	2	2	3	3	3	3
2	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
3	3	3	3	3	2	2	2	2	4	4	4	4	1	1	1	1
4	3	1	4	2	3	1	4	2	4	3	2	1	4	3	2	1
5	4	4	4	4	1	1	1	1	3	3	3	3	2	2	2	2
6	2	4	1	3	2	4	1	3	3	1	4	2	3	1	4	2
7	2	2	2	2	3	3	3	3	1	1	1	1	4	4	4	4
8	4	3	2	1	4	3	2	1	2	4	1	3	2	4	1	3

Table 4.17 Design B1 : $(DD)_1(1324)(1432)(DD)_2(1324)(1324)$ for four treatments.

Subject

Period	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII	XIV	XV	XVI
1	1	1	1	1	4	4	4	4	2	2	2	2	3	3	3	3
2	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
3	3	3	3	3	2	2	2	2	4	4	4	4	1	1	1	1
4	4	3	2	1	4	3	2	1	3	4	1	2	3	4	1	2
5	2	2	2	2	3	3	3	3	1	1	1	1	4	4	4	4
6	3	4	1	2	3	4	1	2	2	1	4	3	2	1	4	3
7	4	4	4	4	1	1	1	1	3	3	3	3	2	2	2	2
8	2	1	4	3	2	1	4	3	4	3	2	1	4	3	2	1

Partition B2.

In this partition of any basic design, Group I contains subjects with sequences 11, 12, 13, 14, 31, 32, 33, 34 in periods 1 and 2 : Group II contains the remaining 8 subjects.

Twelve basic design classes are divided into 3 sets as follows:

Set 1 : DD, BB, DC, DA

Set 2 : CC, CD, CB, CA

Set 3 : AA, AD, AC, AB .

First designs of complex design classes are obtained by performing Combination W (defined in subsection 4.6.2) on these 3 sets. Additional First designs of complex design classes are obtained by performing Operation QB (the 2 odd-numbered periods are periods 3 and 7) and Operation RB.

Partition B3.

In this partition of any basic designs, subjects with sequences 11, 12, 13, 14, 21, 22, 23, 24 in periods 1 and 2 constitute Group I : the other 8 subjects constitute Group II.

Twelve basic design classes are divided into 3 sets as follows:

Set 1 : DD, AA, DC, DB

Set 2 : CC, CD, CB, CA

Set 3 : BB, BD, BC, BA .

First designs of complex design classes are obtained by performing Combination W on these 3 sets. Additional First designs of complex design classes are obtained by performing Operation QB (the 2 odd-numbered periods are periods 5 and 7) and Operation RB.

4.8 Construction of designs of System C.

In this final section, we construct designs of System C for three and four treatments.

4.8.1 Designs for three treatments.

There are only four designs for three treatments in System C. These are the Berenblut designs (1964). Therefore, the four designs by Berenblut (1964) for three treatments are the only designs that are common to all three systems A, B and C.

4.8.2 Designs for four treatments.

There are new designs for four treatments in System C. We have already seen in subsection 4.5.3 that the design in Table 4.20 belongs to System C. However, from the representation of System C designs in Table 4.8, the general construction of all System C designs is not at all obvious. Therefore, we shall consider below a special case of System C designs. The special case of System C designs for four treatments can be written as in Table 4.18.

Table 4.18 Special case of System C designs for four treatments.

	Group			
Period	1	2	3	4
1	I	I	I	I
2	P	Q	R	S
3	A_1	A_1	B_1	B_1
4	A_2P	A_2Q	B_2R	B_2S
5	A_3	A_3	B_3	B_3
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

All the matrices in Table 4.18 are permutation matrices. These matrices have to satisfy certain conditions. To get all combinations in periods 1 and 2, we require

$$P + Q + R + S = J_{(4)} \qquad (4.14)$$

To get all combinations in periods 2 and 3, we require

$$A_1^{-1}(P + Q) + B_1^{-1}(R + S) = J_{(4)} \quad . \tag{4.15}$$

But $$A_1^{-1}(P + Q + R + S) = J_{(4)} \quad ,$$

and $$B_1^{-1}(P + Q + R + S) = J_{(4)} \quad .$$

Hence $$A_1^{-1}(P + Q) = B_1^{-1}(P + Q) \quad .$$

Therefore $$P + Q = A_1B_1^{-1}(P + Q) = B_1A_1^{-1}(P + Q) \quad . \tag{4.16}$$

The possible solutions of (4.16) are

(a) $A_1 = B_1$. This solution is satisfied by designs of Berenblut (1964) and Patterson (1970).

(b) $A_1B_1^{-1}P = Q$; $A_1B_1^{-1}Q = P$. Therefore $B_1A_1^{-1}P = Q$ and hence $(A_1B_1^{-1} - B_1A_1^{-1})P = 0$. But P is not singular. Therefore, $A_1B_1^{-1} = B_1A_1^{-1}$. Hence $A_1 = B_1$. Again, this solution leads to designs by Berenblut (1964) and Patterson (1970).

(c) $\Pi_1 = P + Q = P^1 + Q^1$ where $P^1 = A_1B_1^{-1}P$, $Q^1 = A_1B_1^{-1}Q$ such that $P^1 \neq P$, $Q^1 \neq P$. In other words, it must be possible to express Π_1 as the sum of two permutation matrices in two different ways. The condition for solution (C) is that $\Pi_1 = P + Q$ must be of the form S_1DS_2 where S_1, S_2 are permutation matrices and D is given by

$$D = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad .$$

Proof: Now $P + Q = P^1 + Q^1$. Premultiply both sides by P^{-1} . Therefore, $I_{(4)} + P^{-1} Q = P^{-1} P^1 + P^{-1} Q^1$ such that $P^{-1} P^1$ and $P^{-1} Q^1$ are permutation matrices not equal to $I_{(4)}$. Now a 4×4 permutation matrix has either 0, 1, 2 or 4 unit diagonal elements. The only 4×4 permutation matrix with more than 2 unit diagonal elements is $I_{(4)}$. Therefore, $P^{-1} P^1$ must have 2 unit diagonal elements and $P^{-1} Q^1$ must have 2 unit diagonal elements in the other positions. Hence, there is a permutation matrix R_* such that

$$R_*^{-1} P^{-1} P^1 R_* = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} , \quad R_*^{-1} P^{-1} Q^1 R_* = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} .$$

Write $S_1 = R_*^{-1} P^{-1}$, $S_2 = R_*$. Hence the required condition.

We can show that if $\Pi_1 = P + Q$ can be expressed as the sum of two permutation matrices in two different ways, then so can $\Pi_2 = R + S$, called the <u>complementary matrix</u> of Π_1 . Now, $\Pi_2 = J_{(4)} - \Pi_1$. But $\Pi_1 = S_1 D S_2$. Therefore, $\Pi_2 = J_{(4)} - S_1 D S_2 = S_1^* D S_2^*$ for some permutation matrices S_1^* , S_2^* . Hence the result.

There are 3 possible ways of interchanging any two rows of rows 2, 3 and 4 of matrix D and also 3 possible ways of interchanging any two columns of columns 2, 3 and 4 of matrix D . Therefore, there are 9 matrices of the form $S_1 D S_2$ such that each matrix has unit element in cell (1, 1) . Each of these 9 matrices has a complementary matrix also of the form $S_1 D S_2$. Hence, the total number of matrices of the form $S_1 D S_2$ is 18 .

There are 45 possible partitions of the 16 subjects of any basic design into 2 groups of eight subjects such that every treatment occurs twice in any period of a group. The results obtained above suggest that new designs could be obtained from nine of these partitions. These nine partitions correspond to the 18 matrices of the form $S_1 D S_2$. We will show below that new designs can, in fact, be obtained from these 9 partitions. The steps involved in obtaining new designs for a particular partition, called Partition C1 , will be explained in detail below. These steps are similar to those for obtaining designs of Partitions A1 - A3 , B1 - B3 .

Partition C1 .

Partition C1 involves the partitioning of the 16 subjects of any basic design into 2 groups. Subjects with sequences 11 , 22, 33, 44, 14, 23, 32, 41 in periods 1 and 2 constitute Group I : the other 8 subjects constitute Group II .

Twelve basic design classes are divided into 5 sets as follows:

Set 1 : DD, CC, DC, CD

Set 2 : DB, CB

Set 3 : DA, CA

Set 4 : BD, BC

Set 5 : AD, AC .

Complex designs are obtained by performing Combination W on the above 5 sets. (Complex design and Combination W are defined in subsection 4.6.2). Each such complex design obtained will be known as the First design of a complex design class: other designs of the complex design class are obtained from the First design by performing Permutation P (defined in subsection 4.6.2).

We now define Operation Y .

Operation Y : For the First design of each of the basic design classes in any set, and for each of the First designs of complex design classes obtained by performing Combination W (defined in subsection 4.6.2) on the same set, permute two odd-numbered periods and/or two even-numbered periods of its Group II .

Additional complex designs are obtained as follows:

1. For Set 1, perform Operation Y (permuting periods 3 and 5, periods 4 and 6, and both).
2. For Set 2 and for Set 3, perform Operation Y (permuting periods 3 and 5).
3. For Set 4 and for Set 5, perform Operation Y (permuting periods 4 and 6).

Each complex design obtained will be known as the First design of a complex design class: other designs of the complex design class being obtained from it by performing Permutation P .

Partitions C2 - C9 .

We set out in Table 4.19 the steps involved in obtaining complex designs from each of the Partitions C1 - C9 . (The steps for Partition C1 , already explained in greater detail above, are also given for completeness).

In Table 4.19 , column 1 gives the names of the nine partitions. They are called Partitions C1, C2, ..., C9 . In column 2, the combinations in periods 1 and 2 of the 8 subjects that constitute Group I are given for each partitions: the other 8 subjects constitute Group II . For example, the combinations in periods 1 and 2 of the 8 subjects that constitute Group I of Partition C2 are

period	combination							
1	1	2	3	4	1	2	3	4
2	1	2	4	3	3	4	2	1 .

For each partition, basic design classes are divided into sets. For each of the Partitions C1 , C6 and C7 , twelve basic design classes are divided into 5 sets. For each of the other 6 partitions, eight basic design classes are divided into 3 sets. The sets of basic design classes for each partition are given in column 3 of Table 4.19 . For example, for Partition C4 , the 3 sets of basic design classes are as follows:

Set 1 : DD, DC, AD, AC

Set 2 : DB, AB

Set 3 : DA, AA .

First designs of complex design classes are obtained by performing Combination W (defined in subsection 4.6.2) on the sets of basic design classes of each partition.

Additional First designs of complex design classes are obtained by performing Operation Y on each set of basic design classes of each partition: the 2 odd-numbered periods permuted and/or the 2 even-numbered periods permuted are given in column 4 of Table 4.19 against each set. For example, in Partition C8 , additional First designs of complex design classes are obtained as follows:

1. For Set 1, perform Operation Y (permuting periods 3 and 7, periods 6 and 8, and both).
2. For Set 2 and for Set 3, perform Operation Y (permuting periods 3 and 7).

Complex designs obtained from Partitions C1 - C9 are represented in the same way as those from Partitions A1, A2, A3 and Partitions B1 , B2, B3 . (See subsection 4.6.2). Thus, for example, the design in Table 4.20 is C4 : $(AD)_1(1423)(1342)(AC)_2(1324)(1243)$. This design is obtained from Partition C4 by combining Group I of basic design

AD(1423)(1342) and Group II of basic design AC(1324)(1243) .

Some designs obtained from Partitions C1 - C9 are R-orthogonal and some are not R-orthogonal. For example, the design in Table 4.20 is R-orthogonal. The design in Table 4.21 constructed from Partition C5 is not R-orthogonal.

For any basic design, Group I or Group II of any of the nine partitions can be further subdivided into 2 subgroups of 4 subjects such that every treatment level occurs in period 1 and every treatment level also occurs in period 2 of a subgroup. This result follows by direct application of a theorem given by Ryser (Combinatorial Mathematics, 1963, Chapter 5, Theorem 1.1). By repeated application of this theorem, every treatment level occurs in any period of a subgroup. Therefore, complex designs obtained from Partitions C1 - C9 belong to the special case of System C in Table 4.18 and hence to System C .

Table 4.19 Nine partitions that give complex designs.

Partition	Period	Combinations in periods 1 and 2 of Group I	Sets	Perform Operation Y on periods
C1	1	1 2 3 4 1 2 3 4	1 : DD, CC, DC, CD	3 5, 4 6, both
	2	1 2 3 4 4 3 2 1	2 : DB, CB	3 and 5
			3 : DA, CA	3 and 5
			4 : BD, BC	4 and 6
			5 : AD, AC	4 and 6
C2	1	1 2 3 4 1 2 3 4	1 : DD, DB, CD, CB	3 5, 4 8, both
	2	1 2 4 3 3 4 2 1	2 : CC, DC	3 and 5
			3 : DA, CA	3 and 5
C3	1	1 2 3 4 1 2 3 4	1 : DD, DA, CD, CA	3 5, 6 8, both
	2	1 3 4 2 2 4 3 1	2 : CC, DC	3 and 5
			3 : DB, CB	3 and 5

Table 4.19 (Continued)

Partition	Period	Group I	Sets	Perform Operation Y on periods
C4	1	1 2 3 4 1 2 3 4	1 : DD, DC, AD, AC	5 7, 4 6, both
	2	1 4 2 3 4 1 3 2	2 : DB, AB	5 and 7
			3 : DA, AA	5 and 7
C5	1	1 2 3 4 1 2 3 4	1 : DD, DC, BD, BC	3 7, 4 6, both
	2	1 2 4 3 4 3 1 2	2 : DB, BB	3 and 7
			3 : DA, BA	3 and 7
C6	1	1 2 3 4 1 2 3 4	1 : DD, DB, BD, BB	4 8, 3 7, both
	2	1 2 3 4 3 4 1 2	2 : DC, BC	3 and 7
			3 : DA, BA	3 and 7
			4 : CD, CB	4 and 8
			5 : AD, AB	4 and 8
C7	1	1 2 3 4 1 2 3 4	1 : DD, AA, DA, AD	5 7, 6 8, both
	2	1 2 3 4 2 1 4 3	2 : DC, AC	5 and 7
			3 : DB, AB	5 and 7
			4 : CD, CA	6 and 8
			5 : BD, BA	6 and 8
C8	1	1 2 3 4 1 2 3 4	1 : DD, DA, BD, BA	3 7, 6 8, both
	2	1 3 2 4 2 4 1 3	2 : DC, BC	3 and 7
			3 : DB, BB	3 and 7
C9	1	1 2 3 4 1 2 3 4	1 : DD, DB, AD, AB	5 7, 4 8, both
	2	1 3 2 4 3 1 4 2	2 : DC, AC	5 and 7
			3 : DA, AA	5 and 7

Table 4.20 R-ortho design C4 : $(AD)_1(1423)(1342)(AC)_2(1324)(1243)$ for four treatments

	Subject															
Period	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII	XIV	XV	XVI
1	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
2	1	4	2	3	4	1	3	2	2	3	1	4	3	2	4	1
3	4	3	1	2	4	3	1	2	3	4	2	1	3	4	2	1
4	3	2	4	1	2	3	1	4	4	1	2	3	1	4	3	2
5	2	1	4	3	2	1	4	3	2	1	4	3	2	1	4	3
6	4	1	3	2	1	4	2	3	3	2	4	1	2	3	1	4
7	3	4	2	1	3	4	2	1	4	3	1	2	4	3	1	2
8	2	3	1	4	3	2	4	1	1	4	3	2	4	1	2	3

Table 4.21 Design C5 : $(BD)_1(1324)(1423)(BC)_2(1324)(1432)$ for four treatments.

	Subject															
Period	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII	XIV	XV	XVI
1	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
2	1	2	4	3	4	3	1	2	2	1	3	4	3	4	2	1
3	3	4	1	2	3	4	1	2	3	4	1	2	3	4	1	2
4	4	3	1	2	1	2	4	3	3	4	2	1	2	1	3	4
5	2	3	4	1	2	3	4	1	4	1	2	3	4	1	2	3
6	2	1	3	4	3	4	2	1	1	3	4	2	4	2	1	3
7	4	1	2	3	4	1	2	3	2	3	4	1	2	3	4	1
8	3	4	2	1	2	1	3	4	4	2	1	3	1	3	4	2

Chapter Five

Choice of Serial Factorial Designs

5.1 Introduction.

In this chapter, we consider the choice of suitable SF designs under the following conditions:

1) the treatments of each period are three or four equally spaced levels of a single quantitative factor.
2) the treatment factor, again at three or four levels, is qualitative.
3) the number of treatments is four and they are the treatment combinations of two factors at two levels each.

In each case we use the general model described in Chapter Three to estimate direct × residual interaction components, where appropriate, as well as direct effects and residual effects.

5.2 R-ortho designs.

Throughout this chapter we will restrict ourselves to the R-ortho designs of Systems A, B and C (see Chapter Four). An R-ortho design is defined in subsection 3.6.1 . Further restrictions, where necessary, will be imposed in the later sections. In these designs residual effects are orthogonal, not only to direct effects, but also to direct × residual interactions. Choice of an R-ortho design therefore ensures that residual effects are estimated with as small a variance as possible and without bias due to the existence of unsuspected direct × residual interaction components. The orthogonality also facilitates tests of significance of the interactions themselves.

5.3 Treatments are the equally spaced levels of a single quantitative factor.

In section 5.3 we consider the first case described in section 5.1 ; that is, the treatments are the equally spaced levels of a single quantitative factor. We are only concerned with designs for three and four treatments.

For $t=3$ treatments, we consider all R-ortho designs of Systems A, B and C.

For $t=4$ treatments, we consider

(i) the designs described by Berenblut (1964) and Patterson (1970),

(ii) the designs of System A (all R-orthogonal) constructed by Partitions A1, A2 and A3. (See subsection 4.6.2).

(iii) the R-ortho designs of System B constructed by Partitions B1, B2 and B3. (See subsection 4.7.2).

(iv) the R-ortho designs of System C constructed by Partitions C1, C2, ..., C9. (See subsection 4.8.2).

5.3.1a A design criterion.

We have seen that any direct effect is estimated with full efficiency in all R-ortho designs of System A, B and C. (See Theorems 3.1, 3.4 and 3.6). Also the efficiency factor in the estimation of any residual effect is invariant in these designs (see Theorem 3.7); the efficiency factor is, in fact, $1-\frac{1}{2t(t-1)}$ (see section 3.8). However, the efficiency in the estimation of any component of direct × residual interaction varies in these designs. But the linear direct × linear residual interaction is likely to be most important. Therefore, one criterion for a suitable design among these R-ortho designs is the high efficiency in the estimation of linear direct × linear residual interaction. (Other design criteria will be examined in subsection 5.3.2).

We will only consider suitable designs for three and four treatments. Patterson (1970) showed in the case of four treatments that some basic designs are preferable to others in the estimation of the linear direct × linear residual interaction. (The definition of basic designs is given in subsection 4.6.2) . In subsection 5.3.1d, we extend the comparison to include other R-ortho designs of Systems A, B and C . Choice of suitable designs for three treatments will be considered in subsection 5.3.1c . But first, in the next subsection 5.3.1b , we examine a modified version of the general model described in section 3.3 .

5.3.1b A model for quantitative treatments.

The model can be expressed as

$$E\,y_1 = \mu_1\, 1_{(n)} + \frac{1}{t}\,(X_1 \otimes 1^T_{(t)})\, Q_0^T\,(Q_0\,\alpha) + \beta$$

$$E\,y_i = \mu_i\, 1_{(n)} + X_{i,i-1}\, Q_1^T\,(Q_1\,\alpha) + \beta \qquad i = 2, 3, .., p . \tag{5.1}$$

Q_1 is a $5 \times t^2$ matrix of coefficients of the normalised contrasts estimating linear and quadratic direct effects, linear and quadratic residual effects, and linear direct × linear residual interaction. For example, for $t = 3$ treatments,

$$Q_1 = \begin{bmatrix} -1/\sqrt{6} & -1/\sqrt{6} & -1/\sqrt{6} & 0 & 0 & 0 & 1/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \\ \sqrt{2}/6 & \sqrt{2}/6 & \sqrt{2}/6 & -\sqrt{2}/3 & -\sqrt{2}/3 & -\sqrt{2}/3 & \sqrt{2}/6 & \sqrt{2}/6 & \sqrt{2}/6 \\ -1/\sqrt{6} & 0 & 1/\sqrt{6} & -1/\sqrt{6} & 0 & 1/\sqrt{6} & -1/\sqrt{6} & 0 & 1/\sqrt{6} \\ \sqrt{2}/6 & -\sqrt{2}/3 & \sqrt{2}/6 & \sqrt{2}/6 & -\sqrt{2}/3 & \sqrt{2}/6 & \sqrt{2}/6 & -\sqrt{2}/3 & \sqrt{2}/6 \\ 1/2 & 0 & -1/2 & 0 & 0 & 0 & -1/2 & 0 & 1/2 \end{bmatrix} .$$

Q_0 is a $5 \times t^2$ matrix with the elements in the first two rows the same as Q_1 ; the elements in the other three rows are all zero since no estimates of residual effects or direct × residual interaction are available for the first period. $Q_1\,Q_1^T = I_{(5)}$ and $Q_0\,Q_0^T$ is a 5×5

matrix with 1 in cell (1, 1) and cell (2, 2), and zero in all other cells. $Q_1^T Q_1$ and $Q_0^T Q_0$ are idempotent matrices. The other terms have already been defined in section 3.3.

The normal equations are

$$Q_0 X_{10}^T y_1 + Q_1 \sum_{i=2}^{p} X_{i,i-1}^T y_i = Q_0 X_{10}^T 1_{(n)} \mu_1 + Q_1 \sum_{i=2} X_{i,i-1}^T \mu_i 1_{(n)}$$

$$+ Q_0 X_{10}^T X_{10} Q_0^T (Q_0 \hat{\alpha}) + Q_1 \sum_{i=2} X_{i,i-1}^T X_{i,i-1} Q_1^T (Q_1 \hat{\alpha})$$

$$+ (Q_0 N_0 + Q_1 N) \hat{\beta} \tag{5.2}$$

$$\sum_{i=1} y_i = \sum_{i=1} \mu_i 1_{(n)} + N_0^T Q_0^T (Q_0 \hat{\alpha}) + N^T Q_1^T (Q_1 \hat{\alpha}) + p \hat{\beta}$$

where $N_0 = X_{10}^T$, $N = \sum_{i=2} X_{i,i-1}^T$, p is the number of periods.

The information matrix eliminating subjects is

$$C_Q = Q_0 X_{10}^T X_{10} Q_0^T + Q_1 \sum_{i=2} X_{i,i-1}^T X_{i,i-1} Q_1^T - \frac{1}{p} (Q_0 N_0 + Q_1 N)(N_0^T Q_0^T + N^T Q_1^T) . \tag{5.3}$$

Therefore the variance-covariance matrix is $V_Q = C_Q^{-1} \sigma^2$. Let cell (i, j) of $C_Q^{-1} = v_{ij}$. Then the efficiency factor in the estimation of linear direct × linear residual interaction is ELL_{t^2} given by

$$ELL_{t^2} = \frac{1}{(2t-1)\, v_{55}} .$$

Now V_Q is a diagonal matrix in all R-ortho designs. Also the values of v_{ii}, $i = 1, 2, \ldots, 4$ are invariant, that is, the same for each R-ortho design. In fact, we have

$$v_{11} = v_{22} = 1/2t$$

$$v_{33} = v_{44} = \frac{1}{(2t-1-\frac{1}{2t})} .$$

In contrast, the values of v_{55} and hence ELL_{t^2} vary widely. That is, some designs are more efficient than others in the estimation of linear direct × linear residual interaction. Therefore those R-ortho designs with high value of ELL_{t^2} are suitable designs unless direct and residual effects are known to be additive.

5.3.1c. Designs for three quantitative treatments.

In this subsection, we consider the choice of designs for three treatments. The criterion for a suitable design is the high value of the efficiency factor ELL_9.

The sixteen designs of System A for three treatments are all R-ortho designs. Systems B and C do not yield any additional R-ortho designs. (See Chapter Four, subsections 4.7.1 and 4.8.1). For these 16 designs, ELL_9 has the same value 0.867. All the designs of Table 4.11 are, therefore, equally suitable.

The System B designs that are not R-orthogonal have the value of ELL_9 less than 0.867, for example, the design in Table 4.14. This is because the quadratic residual effect is correlated with the linear direct × linear residual interaction in all these designs. This is one reason why we only examine R-ortho designs.

5.3.1d. Designs for four quantitative treatments.

In this subsection, we consider the choice of suitable designs for four treatments with high efficiency factor ELL_{16}. As stated in the main section 5.3, in the case of four treatments, we are restricting ourselves to the designs by Berenblut (1964) and Patterson (1970) and to the R-ortho designs of Systems A, B and C constructed by the methods described in subsections 4.6.2, 4.7.2 and 4.8.2.

Patterson (1970, Table 6) gave a set of basic designs with $ELL_{16} > 0.95$. Seven of these designs have largest $ELL_{16} = 0.976$.

They are

1. DD(1423)(1324)
2. DD(1324)(1423)
3. DD(1423)(1423)
4. DA(1423)(1324)
5. DA(1423)(1423)
6. AD(1324)(1423)
7. AD(1423)(1423) .

Design DD(1423)(1324) is given in Table 4.1 .

The methods described in subsections 4.6.2 , 4.7.2 and 4.8.2 of this thesis give several new designs with the value of ELL_{16} greater than 0.976 . Some of these new designs are in System A , some are in System B and some are in System C . For example, the System A design in Table 4.13 has $ELL_{16} = 0.991$; the System B design in Table 4.16 and the System C design in Table 4.19 both have $ELL_{16} = 0.982$. The value 0.991 of ELL_{16} is the maximum obtained by R-ortho designs constructed by the methods described in subsections 4.6.2 , 4.7.2 and 4.8.2 . This value is larger than the efficiency factor in the estimation of any residual effect contrast, 0.982 . Besides the design in Table 4.13 , there are two other designs with $ELL_{16} = 0.991$. They are

$$A1 : (DD)_1(1423)(1342)(DD)_2(1234)(1342) ,$$
$$A1 : (DD)_1(1423)(1243)(DD)_2(1234)(1342) .$$

Therefore, all the three designs with maximum $ELL_{16} = 0.991$ belong to System A and are constructed by Partition A1 (see subsection 4.6.2) .

Designs with $ELL_{16} > 0.985$ are given in Appendix 5.1 at the end of the chapter. Designs with values of ELL_{16} in the range 0.980 - 0.985 are given in Appendix 5.2 also at the end of this chapter.

Hence the designs given in Appendix 5.1 and Appendix 5.2 are preferable to the basic designs given by Patterson (1970, Table 6) if the criterion for a suitable design is the high value of ELL_{16}. The value of ELL_{16} is, however, not the only criterion for a suitable design. Other design criteria will be examined in the next subsection 5.3.2.

5.3.2 Other design criteria.

We now consider some other design criteria. In many applications, blocking of subjects is important. Choice of suitable designs for blocking will be considered in the following four subsections 5.3.3a, 5.3.3b, 5.3.3c and 5.3.3d.

Among designs with high values of ELL_{t^2} we will obviously prefer those for which estimates of linear direct × linear residual interaction (LL) are uncorrelated with quadratic direct × linear residual interaction (QL). The interaction component QL is probably the next most important effect not included in the model (5.1). By suitable choice of designs (see subsections 5.3.4a, 5.3.4b and 5.3.4c) we can ensure that a real QL component does not bias the estimate of component LL.

There may also be a trend in time to be eliminated, in which case the error model will not necessarily be appropriate. We would obviously prefer designs with minimum loss of information on interaction component LL due to removing trend. However, this is outside the scope of the present thesis.

Furthermore, some designs have advantages over others when fewer than 2t periods are analysed. Again this aspect will not be considered further in this thesis.

5.3.3a A model with blocking of subjects.

A modified version of model (5.1), allowing for blocking of subjects, can be expressed as

$$E\ y_1 = \mu_1 1_{(n)} + \frac{1}{t}(X_1 \otimes 1^T_{(t)})\, Q_0^T (Q_0\, \alpha) + \beta + Z\gamma \qquad (5.4)$$

$$E\ y_i = \mu_i 1_{(n)} + X_{i,i-1}\, Q_1^T (Q_1\, \alpha) + \beta + Z\gamma$$

where b = number of blocks , Z is an $n\times b$ matrix with cell $(i, j) = 1$ if subject i is in block j, $= 0$ otherwise. γ is the $b\times 1$ column vector of block effects.

Let $\underset{(n\times n)}{S} = I_{(n)} - Z(Z^T Z)^{-1} Z^T$, a symmetric idempotent matrix .

Premultiply (5.4) throughout by S. Since $SZ = 0$, we have

$$E\,S\,y_1 = \mu_1 S 1_{(n)} + S\frac{1}{t}(X_1 \otimes 1^T_{(t)})\, Q_0^T (Q_0 \alpha) + S\beta$$

$$(5.5)$$

$$E\,S\,y_i = \mu_i S 1_{(n)} + S X_{i,i-1}\, \alpha + S\beta$$

where $Z^T y_i$ is a $b\times 1$ column vector of block totals in period i.

The error model is now $\mathrm{var}(S_0\, y) = (I_{(p)} \otimes S)\,\sigma^2$, where $S_0 = \underset{p \text{ terms}}{\text{diagonal}(S\ S \dots S)}$ and $y = \begin{pmatrix} y_1 \\ \vdots \\ y_i \\ \vdots \\ y_p \end{pmatrix}$.

We will only be concerned with equal-sized blocks. That is, the n subjects are divided into b blocks of n/b subjects each, n/b an integer. Therefore, $S = I_{(n)} - \frac{b}{n} Z Z^T$.

The normal equations for estimating α and β are

$$Q_0 X_{10}^T S y_1 + Q_1 \sum_{i=2} X_{i,i-1}^T S y_i = \mu_1 Q_0 X_{10}^T S 1_{(n)} + Q_1 \sum_{i=2} X_{i,i-1}^T \mu_i S 1_{(n)}$$

$$+ Q_0 X_{10}^T S X_{10} Q_0^T (Q_0 \hat{\alpha})$$

$$+ Q_1 \sum_{i=2} X_{i,i-1} S X_{i,i-1} Q_1^T (Q_1 \hat{\alpha})$$

$$+ (Q_0 N_0 + Q_1 N) S \hat{\delta} \qquad (5.6)$$

$$\sum_{i=1} S y_i = S X_{10} Q_0^T (Q_0 \hat{\alpha}) + \sum_{i=2} S X_{i,i-1} Q_1^T (Q_1 \hat{\alpha}) + p S \hat{\delta}$$

where $N_0 = X_{10}^T$, $N = \sum_{i=2} X_{i,i-1}^T$, p is the number of periods.

The information matrix is therefore given by

$$\underset{(5\times5)}{C_{QB}} = Q_0 X_{10}^T S X_{10} Q_0^T + Q_1 \sum_{i=2} X_{i,i-1}^T S X_{i,i-1} Q_1^T$$

$$- \frac{1}{p} (Q_0 N_0 + Q_1 N) S (Q_0 N_0 + Q_1 N)^T . \qquad (5.7)$$

The terms on the right-hand side of (5.7) can be expanded, giving

$$\underset{(5\times5)}{C_{QB}} = Q_0 X_{10}^T X_{10} Q_0^T + Q_1 \sum_{i=2} X_{i,i-1}^T X_{i,i-1} Q_1^T - \frac{b}{n} Q_0 X_{10}^T Z Z^T X_{10} Q_0^T$$

$$- \frac{b}{n} Q_1 \sum_{i=2} X_{i,i-1}^T Z Z^T X_{i,i-1} Q_1^T - \frac{1}{p} (Q_0 N_0 + Q_1 N)(Q_0 N_0 + Q_1 N)^T$$

$$+ \frac{b}{pn} (Q_0 N_0 + Q_1 N) Z Z^T (Q_0 N_0 + Q_1 N)^T \qquad (5.8)$$

Therefore (5.8) differs from the information matrix for the model without blocking of subjects in (5.3) by the three terms involving $Z Z^T$.

The variance-covariance matrix is then $V_{QB} = C_{QB}^{-1} \sigma^2$. Let cell (i, j) of $C_{QB}^{-1} = u_{ij}$. Then the efficiency factor in the estimation of the linear direct × linear residual interaction of a design arranged in blocks of t^2/b subjects is $ELL_{t^2/b}$ given by

$$ELL_{t^2/b} = \frac{1}{(2t-1)u_{55}} .$$

We have previously shown that the variance-covariance matrix of any R-ortho design with subjects not arranged in blocks is a diagonal matrix. When the subjects are blocked, the variance-covariance matrix V_{QB} is sometimes but not always a diagonal matrix. When direct effects and residual effects are not confounded with blocks, V_{QB} is diagonal (even though the linear direct × linear residual interaction may be confounded).

Consider, for example, the design for three treatments by Berenblut (1964) in Table 2.8. This design can be arranged in blocks (block 1 = subject I, II, III; block 2 = subjects IV, V, VI; block 3 = subjects VII, VIII, IX) such that no direct effects or residual effects are confounded with blocks. In this case, the matrix V_{QB} is diagonal even though any component of direct × residual interaction is confounded.

In contrast, the subjects of the System A design for three treatments given in Table 4.9 cannot be arranged in blocks of 3 subjects without confounding direct and residual effects with blocks. This result also applies to any other design in System A that is not also in System C. That is, the only designs for three treatments that are suitable for blocking are the designs described by Berenblut (1964).

It is always possible to arrange an R-ortho design of System C in blocks of t subjects without confounding direct effects or residual effects. Hence V_{QB} is diagonal. Each block is given by a group of t subjects as shown in Table 4.8. In the case $t=4$, the designs can also be arranged in blocks of $2t$ subjects by combining pairs of groups. It follows that all the R-ortho designs for which V_{QB} is not diagonal must be in System A or System B. Some System A and System B designs for four treatments can, however, be efficiently arranged in blocks of four or blocks of eight. Others can be arranged in blocks of eight but not blocks of four; an example is provided by the System A design in Table 4.13.

In the following three subsections we consider the problem of arranging the t^2 subjects of R-ortho designs in b blocks of $t^2/_b$ subjects in such a way that

a) direct and residual effects are not confounded, that is, V_{QB} is a diagonal matrix with

$$u_{ii} = v_{ii} \quad \text{(see subsection 5.3.1b)} \quad i = 1, \ldots, 4$$

b) $ELL_{t^2/_b} = ELL_{t^2}$ or $ELL_{t^2/_b}$ is as large as possible.

<u>5.3.3b Designs for three treatments in blocks of three subjects.</u>

In this subsection, we consider the choice of designs for three treatments arranged in blocks of three subjects. The criteria for a suitable design are that direct and residual effects are not confounded with blocks and the efficiency factor ELL_3 is as large as possible.

All the sixteen designs of System A for three treatments are R-orthogonal: these include the four designs by Berenblut (1964). Systems B and C do not yield any additional R-ortho designs.

The twelve new designs of System A cannot be arranged in blocks of three subjects without confounding direct and residual effects. These designs are, therefore, not suitable for blocking. The four designs by Berenblut (1964) can, however, be so arranged without confounding direct and residual effects. But such blocking results in serious loss of information on linear direct × linear residual interaction; that is, the value of ELL_3 is very low compared to ELL_9. For example, the design by Berenblut (1964) in Table 2.8 can be arranged in blocks of three subjects as follows:

block 1 : subjects I - III ,

block 2 : subjects IV - VI ,

block 3 : subjects VII - IX .

Then the value of the efficiency factor ELL_3 is 0.483. Thus, no designs for three treatments are suitable for blocking unless direct and residual effects are known to be additive, in which case the four designs by Berenblut (1964) can be used.

5.3.3c Designs for four treatments in blocks of eight subjects.

In this subsection, we will consider the choice of suitable designs for four treatments arranged in blocks of eight subjects. The criteria for a suitable design are that direct and residual effects are not confounded with blocks and the efficiency factor ELL_8 is equal to ELL_{16} (the efficiency factor for the same design that is not blocked) and is as large as possible.

We have already stated in subsection 5.3.3a that all R-ortho designs of System C for four treatments can be arranged in blocks of eight subjects without confounding direct and residual effects. Now some System C designs constructed by Partitions C1, C2, ..., C9 are R-orthogonal and can, therefore, be arranged in blocks of eight subjects without confounding direct and residual effects. The R-ortho designs constructed by Partitions C1, C2, ..., C6 have an additional property in that each of these designs can be arranged in blocks of eight subjects such that the efficiency factors ELL_8 and ELL_{16} are the same. An example is provided by the R-ortho design in Table 4.20 constructed by Partition C4. In this design, Subjects I - VIII constitute block 1 ;

the other eight subjects constitute block 2 . The R-ortho designs constructed by Partitions C7, C8 and C9 cannot be efficiently blocked such that $ELL_8 = ELL_{16}$.

Except for some designs in basic design classes AA, BB and AB, all designs constructed by Patterson (1970) for four treatments can be efficiently blocked. (See Patterson (1970)). For example, the design in Table 4.1 can be efficiently blocked. Subjects I - VIII constitute block 1 ; the other eight subjects constitute block 2.

It is not at all obvious which designs of System A and R-ortho designs of System B have the property that they can be arranged in blocks of eight subjects such that direct and residual effects are not confounded. It is even more difficult to determine which of the designs satisfying the above property also have $ELL_8 = ELL_{16}$.

We will, therefore, only examine the designs in Appendix 5.1 and Appendix 5.2, for which $ELL_{16} > 0.980$, on their suitability for blocking. All the designs in these appendices can be arranged in blocks of eight subjects without confounding direct and residual effects. However, only those designs marked with an asterisk * can be arranged such that $ELL_8 = ELL_{16}$. These designs are, therefore, suitable. For example, the System A design in Table 4.13 is suitable. Subjects I, IV, V, VIII, X, XI, XIV, XV constitute block 1 ; the other 8 subjects constitute block 2 . Direct effects and residual effects are then not confounded with blocks and $ELL_8 = ELL_{16} = 0.991$. So also are the other two designs in Appendix 5.1 for which $ELL_8 = ELL_{16} = 0.991$. The System B design in Table 4.16 and the System C design in Table 4.20 are also suitable.

5.3.3d Designs for four treatments in blocks of four subjects.

In this subsection, we consider the choice of suitable designs for four treatments arranged in blocks of four subjects. The criteria for a suitable design are that direct effects and residual effects are not confounded with blocks and the efficiency factor ELL_4 is as large as possible. Patterson (1970) showed that no basic design exists such that $ELL_4 = ELL_{16}$. This result can easily be extended to any SF design for four treatments.

It is obvious, again, that all R-ortho designs of System C (including designs by Berenblut (1964) and Patterson (1970)) for four treatments can be arranged in blocks of four subjects without confounding direct and residual effects. Some designs of System A and some R-ortho designs of System B can also be so arranged. It is obvious that these designs also belong to System C.

We first examine the designs in Appendix 5.1 and Appendix 5.2, for which $ELL_{16} > 0.980$, on their suitability for this blocking. The designs in these appendices cannot be arranged in blocks of four subjects without confounding direct effects and residual effects, or such that $ELL_4 > 0.90$. For example, the design in Table 4.13 with $ELL_{16} = 0.991$ cannot be arranged in blocks of four subjects without confounding direct and residual effects. Hence, the designs in these appendices are not suitable for this blocking even though those designs marked with an asterisk * are suitable for arranging in blocks of eight subjects (see subsection 5.3.3c).

Patterson (1970) showed, however, that some basic designs of Set DD can be arranged in blocks of four subjects such that $ELL_4 > 0.90$. The allocation of subjects to blocks to minimize the loss of information on linear direct × linear residual interaction is made by using the following Latin square in transformation set II (see Table 4.4)

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 3 & 4 & 1 & 2 \\ 2 & 1 & 4 & 3 \end{array}$$

Rows represent treatments in period 1, in the order 1 2 3 4, columns represent treatments in period 2 and the numbers are block numbers. For example, the allocation of the basic design by Patterson (1970) in Table 4.1 is as follows:

block 1 : subjects V - VIII block 2 : subjects IX - XII

block 3 : subjects XIII - XVI block 4 : subjects I - IV

The value of ELL_4 of this design is 0.943. Such allocation of subjects to blocks suggest that new R-ortho designs of System C constructed by Partitions C3, C4 and C6 exist such that they can be arranged in blocks of four subjects with $ELL_4 > 0.90$. Using the similar scheme as above for the allocation of subjects to blocks, such designs do, in fact, exist. Those designs with $ELL_4 > 0.920$ are given in Appendix 5.3. Designs by Patterson (1970) that can be arranged in blocks of four subjects with $ELL_4 > 0.920$ are also given in Appendic 5.3. Hence, the designs in Appendix 5.3 are suitable for blocking, with four subjects to a block.

Designs that can be suitably arranged in blocks of eight subjects and also in blocks of four subjects are obviously to be preferred. We have seen earlier in this subsection that the designs in Appendix 5.1 and Appendix 5.2 marked with an asterisk are suitable in blocks of eight subjects and not in blocks of four subjects. All designs in Appendix 5.3 can also be arranged in blocks of eight subjects such that direct and residual effects are not confounded with blocks and $ELL_8 = ELL_{16}$. There are five designs in Appendix 5.3 with $ELL_8 = ELL_{16} = 0.976$. These five designs are, therefore, reasonably suitable in blocks of eight subjects as well. Three of these designs are constructed by Patterson (1970) and belong to basic design class DD: the other two are constructed by Partition C4 (see subsection 4.8.2).

One of the three designs by Patterson (1970) is in Table 4.1 .

5.3.4a Quadratic direct × linear residual interaction (QL) .

Although the estimation of quadratic direct × linear residual interaction (QL) may not be important, designs with linear direct × linear residual interaction (LL) uncorrelated with QL would be preferred since this will ensure that the estimation of LL will not be biased by a real QL component.

In subsection 5.3.4b , we consider the selection of designs for three treatments with QL uncorrelated with LL . Subsection 5.3.4c deals with designs for four treatments. For this purpose, we require $6 \times t^2$ matrices, Q_0^* and Q_1^* , the first five rows of which are given by Q_0 and Q_1 . The sixth row of Q_0^* has all elements zero; the sixth row of Q_1^* is the row vector giving the coefficients of the normalised contrast of quadratic direct × linear residual interaction. For example, for $t = 3$ treatments, the sixth row of Q_1^* is given by

$$\left(-\sqrt{3}/6 \quad 0 \quad \sqrt{3}/6 \quad \sqrt{3}/3 \quad 0 \quad -\sqrt{3}/3 \quad -\sqrt{3}/6 \quad 0 \quad \sqrt{3}/6 \right) .$$

We use the model in (5.1) but with the matrices Q_0 and Q_1 replaced by Q_0^* and Q_1^* . The information matrix eliminating subjects will be represented by C_Q^* and the variance-covariance matrix by V_Q^* where $V_Q^* = C_Q^{*-1}\sigma^2$; also let w_{ij} be the element in cell (i, j) of C_Q^{*-1} . Then, for all R-ortho designs, the interaction component LL is uncorrelated with QL if

$$w_{56} = w_{65} = 0 \quad .$$

5.3.4b Designs for three treatments.

In this subsection, we consider the selection of designs for three treatments. The criteria for a suitable design are that the efficiency factor ELL_9 is large and the interaction component LL is uncorrelated with QL .

All the sixteen R-ortho designs in System A for three treatments have $ELL_9 = 0.867$. (See subsection 5.3.1c). But there are only two designs with LL uncorrelated with QL . They are

$$A_1(132)R_1(123)\ A_2(132)R_2(231)\ A_3(132)R_3(312)\ ,$$
$$A_1(123)R_1(132)\ A_2(123)R_2(213)\ A_3(123)R_3(321)\ .$$

Both designs are constructed by Berenblut (1964) and are the only binary designs of System A. (The definition of a binary design is given in subsection 3.6.3).

5.3.4c Designs for four treatments.

In this subsection, we consider the selection of designs for four treatments. The criteria for a suitable design are that the efficiency factor ELL_{16} is large and the interaction component LL is uncorrelated with QL . We will, therefore, only examine the designs in Appendix 5.1 and Appendix 5.2 , for which $ELL_{16} > 0.980$. Those designs in which LL is uncorrelated with QL are suitable. They are marked with + . For example, the System A design in Table 4.13 with $ELL_{16} = 0.991$ and the System B design in Table 4.16 with $ELL_{16} = 0.982$ both have LL uncorrelated with QL , but not the System C design in Table 4.20 with $ELL_{16} = 0.982$. The other two designs in Appendix 5.1 for which $ELL_{16} = 0.991$ are also suitable.

The designs in Appendix 5.1 marked with + are also marked with * . Therefore, these designs are suitable for blocking in blocks of eight subjects or when the interaction component QL is real. Only the two System A designs constructed by Partition A2 with $ELL_{16} = 0.988$ are suitable if arranged in blocks of eight subjects but not suitable if QL is real.

5.4 Treatments are qualitative.

In this main section, we consider the second case stated in section 5.1 ; that is, the treatments are qualitative. We are only concerned with selection of designs for three and four treatments.

For $t = 3$ treatments, we restrict ourselves to all R-ortho designs of Systems A, B and C .

For $t = 4$ treatments, we consider

(i) the designs described by Berenblut (1964) and Patterson (1970).

(ii) the designs of System A (all R-orthogonal) constructed by Partitions A1, A2 and A3 . (See subsection 4.6.2).

We exclude the R-ortho designs of Systems B and C constructed by the methods described in subsections 4.7.2 and 4.8.2 .

5.4.1a A model for qualitative treatments.

The model can be expressed as

$$\begin{aligned} E\, y_1 &= \mu_1\, 1_{(n)} + \frac{1}{t}\,(X_1 \otimes 1^T_{(t)})\, T_1\, \alpha + \beta \\ E\, y_i &= \mu_i\, 1_{(n)} + X_{i,i-1}(T_1 + T_2 + T_{12})\, \alpha + \beta \quad i = 2, 3, \dots, p \end{aligned} \tag{5.9}$$

The matrices T_1, T_2 and T_{12} are defined in Appendix 3.1 . The other terms have already been defined in section 3.3 .

The information matrix on direct effects, residual effects and direct × residual interaction, $(T_1 + T_2 + T_{12})\,\hat{\alpha}$, eliminating subjects, is

$$\begin{aligned} C_{QA} &= T_1\, X^T_{10} X_{10}\, T_1 + (T_1 + T_2 + T_{12}) \sum_{i=2} X^T_{i,i-1}\, X_{i,i-1}\, (T_1 + T_2 + T_{12}) \\ &\quad - \frac{1}{p}\,[T_1\, N_0 + (T_1 + T_2 + T_{12})\, N][T_1\, N_0 + (T_1 + T_2 + T_{12})\, N\,]^T \end{aligned} \tag{5.10}$$

The difference between the general model in (3.14) and the model in (5.9) is that we do not estimate $T_0\alpha$ in (5.9) . (The matrix T_0 is defined in Appendix 3.1) . The information matrix on $T_0\hat{\alpha}$ for the model in (3.14) is $T_0 C T_0$ where C, given in (3.23), is

$$C = X_{10}^T X_{10} + \sum_2 X_{i,i-1}^T X_{i,i-1} - \frac{1}{p}(N_0 + N)(N_0^T + N^T) \qquad (5.11)$$

Equation (5.10) can be rewritten

$$C_{QA} = (T_1 + T_2 + T_{12})\, C\, (T_1 + T_2 + T_{12}) .$$

But $I_{(t^2)} - T_0 = T_1 + T_2 + T_{12}$ (see Appendix 3.1) .

Hence $C_{QA} = (I_{(t^2)} - T_0)\, C\, (I_{(t^2)} - T_0)$.

Now $$T_0 C = 0 . \qquad (5.12)$$

Therefore, we have $C_{QA} = C$.

__5.4.1b Design criteria.__

The information matrix on direct effects, $T_1 \hat{\alpha}$, is $T_1 C T_1$. But for any SF design $T_1 C = 2t T_1$ (see Theorem 3.4). Therefore

$$T_1 C T_1 = 2t T_1 \qquad (5.13)$$

For R-ortho designs, the information matrix on residual effects, $T_2 \hat{\alpha}$, is $T_2 C T_2$. But $T_2 C = (2t - 1 - \frac{1}{2t})\, T_2$ for all R-ortho designs. (See Theorem 3.5). Therefore,

$$T_2 C T_2 = (2t - 1 - \frac{1}{2t})\, T_2 . \qquad (5.14)$$

For R-ortho designs, the information matrix on the direct × residual interaction, $T_{12} \hat{\alpha}$, is $T_{12} C T_{12}$. Since $T_{12} = I_{(t^2)} - T_0 - T_1 - T_2$, it can be easily shown, by using results in (5.12), (5.13) and (5.14), that $T_{12} C T_{12}$ is equal to C_{QU}, where C_{QU} is given by

$$C_{QU} = C - 2t T_1 - (2t - 1 - \frac{1}{2t})\, T_2 . \qquad (5.15)$$

That is, for an R-ortho design, the information matrix of direct × residual interaction is the same as the information matrix on $\hat{\alpha}$, eliminating subjects, direct effects and residual effects.

The values of N_0 and the first two terms on the right-hand side of equation (5.11) are invariant for any SF design. But the incidence matrix, N, of the associated incomplete block design of an SF design and, therefore, C vary for different designs. Hence the information matrix C_{QU} varies for different R-ortho designs.

A measure of the information on the direct × residual interaction is $\operatorname{tr} C_{QU}$. Now, from section 3.8, we have

$$\operatorname{tr} C = t + t^2(2t-1) - \frac{1}{2t}\,(3t + \operatorname{tr} NN^T) \;.$$

Therefore, $\operatorname{tr} C_{QU} = t(2t^2 - 5t + 6) - \frac{1}{2t}(1 + 4t + \operatorname{tr} NN^T)$.

(Note that this is the same value as x in section 3.8).

Maximum information on direct × residual interaction is attained by binary designs among R-ortho designs. (See subsection 3.8.1). The values of $\operatorname{tr} C_{QU}$ are $17\,^2/_3$ for $t=3$ treatments and $55\,^7/_8$ for $t=4$ treatments.

Other criteria may also be important. Three criteria will be considered. But first, we will give another expression for the information matrix of direct × residual interaction of an R-ortho design: we also define similar designs.

The information matrix of direct × residual interaction of an R-ortho design can also be expressed as

$$C^* = Q\,C\,Q^T \;,$$

where Q is a $(t-1)^2 \times t^2$ matrix of coefficients such that $QQ^T = I$, $Q^T Q = T_{12}$. The variance-covariance matrix is then V^* where $V^* = C^{*-1}\sigma^2$.

Def$^{\underline{n}}$: Let the concurrence matrices of the associated incomplete block (a.i.b) designs of any two SF designs (not necessarily R-ortho and/or binary) be $N_1N_1^T$ and $N_2N_2^T$. If there exists a $t^2 \times t^2$ permutation matrix P such that $N_1N_1^T = P N_2N_2^T P^T$, then the two SF designs are said to be <u>similar designs</u>.

For example, the design in Table 2.8 with the concurrence matrix, NN^T, of its a.i.b. design given in Table 3.3 and design $A_1(132)R_1(132)$ $A_2(132)R_2(213)$ $A_3(132)R_3(321)$ are similar designs.

The concept of similar designs is useful because if $N_1N_1^T = P N_2N_2^T P^T$ (P a permutation matrix), then the matrices $N_1N_1^T$ and $N_2N_2^T$ have the same set of eigenvalues. (See Theorem C.6 in Appendix C). Let the information matrices of direct × residual interaction of the two similar designs be C_1^* and C_2^*. Then, using the fact that the values of N_0 and the first two terms in (5.11) are invariant for any SF design and again applying Theorem C.6, the information matrices C_1^* and C_2^* also have the same set of eigenvalues. Hence, by applying Theorem C.5, the matrices C_1^{*-1} and C_2^{*-1} have the same set of eigenvalues.

We now consider the three additional criteria for a suitable design. These criteria are called optimality criteria.

Let d denote an R-ortho binary design with the information matrix of direct × residual interaction given by C_d^*. The class of d's is denoted by Δ; that is, Δ includes only R-ortho binary designs. The three optimality criteria are given below. (See Kiefer (1958), Kiefer and Wolfowitz (1959) and Kiefer (1959)).

(1) A design d_1 is said to be <u>D-optimum</u> in Δ if $d_1 \in \Delta$ and

$$\det. C_{d_1}^{*-1} = \min_{d\in\Delta} \det. C_d^{*-1}$$

$$= \min_{d\in\Delta} \prod_{i=1}^{(t-1)^2} \lambda_i \text{, by applying}$$

Theorem C.4 of Appendix C, where λ_i, $i = 1, 2, \ldots, (t-1)^2$ are the eigenvalues of C_d^{*-1}. (See Appendix C).

(2) A design d_1 is said to be <u>E-optimum</u> in Δ if $d_1 \in \Delta$ and

$$\lambda_{max}(C_{d_1}^{*-1}) = \min_{d \in \Delta} \lambda_{max}(C_d^{*-1})$$

where $\lambda_{max}(C_d^{*-1})$ is the maximum eigenvalue of C_d^{*-1} .

(3) A design d_1 is said to be <u>A-optimum</u> in Δ if $d \in \Delta$ and

$$\text{trace } C_{d_1}^{*-1} = \min_{d \in \Delta} \text{trace } C_d^{*-1}$$

$$= \min_{d \in \Delta} \sum_{i=1}^{(t-1)^2} \text{ , using Theorem C.4 ,}$$

where λ_i , $i = 1, 2, \ldots, (t-1)^2$ are the eigenvalues of C_d^{*-1} .

Applying Theorem C.3 , we have

$$Z^T C^{*-1} Z = D$$

where $D = \text{diag}\left\{\lambda_i,\ i = 1, \ldots, (t-1)^2\right\}$ and $Z = [Z_1 \ldots Z_{(t-1)^2}]$ is a $(t-1)^2 \times (t-1)^2$ matrix of eigenvectors $\underset{(t-1)^2 \times 1}{Z_i}$, $i = 1, 2, \ldots, (t-1)^2$ corresponding to eigenvalues λ_i , $i = 1, 2, \ldots, (t-1)^2$ of C^{*-1} . Then the variance-covariance matrix of direct × residual interaction contrasts $Z^T Q \hat{\alpha}$ is $D\sigma^2$.

Let Z_{max} be the eigenvector associated with the largest eigenvalue λ_{max} of C_d^{*-1} . Then $Z_{max}^T Q \hat{\alpha}$ has at least as large a variance $\lambda_{max}\sigma^2$ as any other normalised interaction contrast. An E-optimum design keeps this variance at a minimum.

Criteria (1) and (2) can also be expressed in words. An R-ortho binary design is said to be A-optimum in the class of R-ortho binary designs if the average variance of the normalised direct × residual interaction contrasts given by $Z^T Q \hat{\alpha}$ is a minimum. An R-ortho binary design is said to be D-optimum in the class of R-ortho binary design if the generalized variance of $Z^T Q \hat{\alpha}$ is a minimum.

In the next subsection, we consider the selection of designs for three treatments. Subsection 5.4.1d deals with designs for four treatments.

5.4.1c Designs for three qualitative treatments.

All sixteen designs of System A for three treatments are R-orthogonal. There are no additional R-ortho designs in Systems B and C. But only two of these 16 designs are binary. They are

$$A_1(132)R_1(123)\ A_2(132)R_2(231)\ A_3(132)R_3(312)\ ,$$
$$A_1(123)R_1(132)\ A_2(123)R_2(213)\ A_3(123)R_3(321)\ .$$

Both designs are constructed by Berenblut (1964). Let these two designs be d_1 and d_2, and their information matrices of direct × residual interaction be $C^*_{d_1}$ and $C^*_{d_2}$. But these two designs are similar. Therefore, the matrices $C^{*-1}_{d_1}$ and $C^{*-1}_{d_2}$ have the same set of eigenvalues, 0.261, 0.261, 0.207 and 0.207. Hence it is trivial to show that both designs are D-, E- and A-optimum.

We can show, in a non-trivial sense, that these two designs are indeed the best by comparing them with the other 14 R-ortho designs that are not binary. We set out in Table 5.1 the values of λ_{max}, $\sum_{i=1}^{4}\lambda_i$ and $\prod_{i=1}^{4}\lambda_i$ of the two R-ortho binary designs and the 14 R-ortho designs that are not binary. (λ_i, $i = 1, \ldots, 4$ and λ_{max} are defined in subsection 5.4.1b). Hence the two R-ortho binary designs are the best. Note that the new System A designs are better than the two non-binary designs of Berenblut (1964). The main reason is that λ_{max} in these two designs, both similar, has a comparatively large value of 1.20, even though the other three eigenvalues are small and have the same value 0.207. In Table 5.1, designs with the same value of λ_{max} are similar designs.

Table 5.1 Values of λ_{max}, $\sum_{i=1}^{4} \lambda_i$ and $\prod_{i=1}^{4} \lambda_i$ of all 16 designs of System A for three treatments. (Small values of λ_i, $i = 1, \ldots 4$ are required).

Berenblut's designs	λ_{max}	$\sum_{i=1}^{4} \lambda_i$	$\prod_{i=1}^{4} \lambda_i \times 10^3$
$A_1(123)\ R_1(123)\ A_2(123)\ R_2(231)\ A_3(123)\ R_3(312)$	1.20	1.821	10.628
$A_1(132)\ R_1(123)\ A_2(132)\ R_2(231)\ A_3(132)\ R_3(312)$	0.261	0.936	2.913
$A_1(123)\ R_1(132)\ A_2(123)\ R_2(213)\ A_3(123)\ R_3(321)$	0.261	0.936	2.913
$A_1(132)\ R_1(132)\ A_2(132)\ R_2(213)\ A_3(132)\ R_3(321)$	1.20	1.821	10.628
Other System A designs			
$A_1(132)\ R_1(123)\ A_2(132)\ R_2(231)\ A_3(123)\ R_3(312)$	0.292	0.973	3.388
$A_1(123)\ R_1(132)\ A_2(132)\ R_2(213)\ A_3(132)\ R_3(321)$	0.458	1.108	4.640
$A_1(123)\ R_1(123)\ A_2(123)\ R_2(231)\ A_3(132)\ R_3(312)$	0.458	1.108	4.640
$A_1(132)\ R_1(132)\ A_2(123)\ R_2(213)\ A_3(123)\ R_3(312)$	0.292	0.973	3.388
$A_1(132)\ R_1(123)\ A_2(123)\ R_2(231)\ A_3(132)\ R_3(312)$	0.292	0.973	3.388
$A_1(132)\ R_1(132)\ A_2(132)\ R_2(213)\ A_3(123)\ R_3(321)$	0.458	1.108	4.640
$A_1(123)\ R_1(123)\ A_2(132)\ R_2(231)\ A_3(123)\ R_3(312)$	0.458	1.108	4.640
$A_1(123)\ R_1(132)\ A_2(123)\ R_2(213)\ A_3(132)\ R_3(321)$	0.292	0.973	3.388
$A_1(123)\ R_1(123)\ A_2(132)\ R_2(231)\ A_3(132)\ R_3(312)$	0.292	0.973	3.388
$A_1(132)\ R_1(132)\ A_2(123)\ R_2(213)\ A_3(132)\ R_3(321)$	0.458	1.108	4.640
$A_1(132)\ R_1(123)\ A_2(123)\ R_2(231)\ A_3(123)\ R_3(312)$	0.458	1.108	4.640
$A_1(123)\ R_1(132)\ A_2(132)\ R_2(213)\ A_3(123)\ R_3(321)$	0.292	0.973	3.388

One of the two R-ortho binary designs can be obtained from the other by interchanging any two of the three numbers 1, 2, 3 in the designs. In this application in which the treatments are qualitative, besides randomizing the subjects, the treatments are randomly allotted to the numbers 1, 2, 3 in the design. Hence, in a sense, both designs are the same for this application.

The two R-ortho binary designs are also suitable for the first application in section 5.1 in which the criteria for a suitable design are that the efficiency factor $E\,L\,L_9$ is large and the interaction component LL is uncorrelated with QL. (See subsection 5.3.4b).

5.4.1d Designs for four qualitative treatments.

In this subsection, we consider the selection of designs for four treatments. The criteria for a suitable design are that the design is an R-ortho binary design and its values of λ_{max}, $\sum_{i=1}^{9} \lambda_i$ and $\prod_{i=1}^{9} \lambda_i$ (see subsection 5.4.1b) are as small as possible.

As stated in section 5.4, we will only examine the designs by Berenblut (1964) and Patterson (1970) and the designs constructed by Partitions A1, A2 and A3 (see subsection 4.6.2) for their suitability.

The R-ortho binary designs that are good on the three optimality criteria are given in Appendix 5.4 at the end of the chapter. In Appendix 5.4, there are six designs that are D-, E- and A-optimum among the class of R-ortho binary designs examined. These 6 designs are therefore the best designs. All six designs are basic designs constructed by Berenblut (1964); there are two in each of the basic design classes AA, BB and CC. The six designs are

AA(1432)(1342) , BB(1243)(1423) , CC(1324)(1234) ,

AA(1342)(1432) , BB(1423)(1243) , CC(1234)(1324) .

These six designs are also similar designs.

In Appendix 5.4, there are twelve other designs that are E-optimum only. These are all basic designs also constructed by Berenblut (1964); there are four designs in each of the basic design classes AA, BB and CC. These twelve designs are also similar designs.

There are no suitable designs in basic design class DD and mixed design classes constructed by Patterson (1970) (see section 4.3). For example, the design in Table 4.1 is not suitable.

No design of System A constructed by Partitions A1, A2 and A3 is D-, E- or A-optimum. However, there are some R-ortho binary designs that are reasonably good on the three optimality criteria. These designs are given in Appendix 5.4.

It is of interest to consider whether designs that are suitable for qualitative treatments are also useful when treatments are equally spaced levels of a single quantitative factor, and vice versa. The six best designs in Appendix 5.4 are not useful for quantitative treatments. The efficiency factor in the estimation of the linear direct × linear residual interaction given by ELL_{16} (see subsection 5.3.1b) for these designs is less than 0.90. Both designs in basic design class AA have $ELL_{16} = 0.864$; both designs in basic design class BB have $ELL_{16} = 0.885$ and, finally, both designs in basic design class CC have $ELL_{16} = 0.875$.

All designs in Appendix 5.1 for which $ELL_{16} > 0.985$ are R-orthogonal. Except for one binary design marked with a 'b', all other designs in this appendix are not binary. For example, the System A design in Table 4.13 with $ELL_{16} = 0.991$ is not binary. Even the binary design in the appendix is not suitable for qualitative treatments. Therefore, designs that are best for quantitative treatments are not necessarily useful for qualitative treatments.

5.5 The four treatments are treatment combinations of two factors at two levels each.

We now consider the case in which the four treatments are the treatment combinations of two factors, each at two levels. We will restrict ourselves to the designs by Berenblut (1964) and Patterson (1970) and the designs of System A constructed by Partitions A1, A2 and A3.

Let the two factors be A and B. Also, following the usual convention in factorial experiments, let the four treatment combinations (1), a, b, ab correspond to treatments 1, 2, 3, 4 in a design. Then we have

main effect of A $= \frac{1}{2}(a + ab - b - (1))$ denoted by U ,

main effect of B $= \frac{1}{2}(a + ab - a - (1))$ denoted by V ,

AB interaction $= \frac{1}{2}(ab + (1) - a - b)$ denoted by W .

We also need to distinguish between direct and residual effects. This is done by using suffices d and r. Thus U_d, V_d, W_d refer to direct effects; U_r, V_r, W_r to residual effects. Also $U_d V_r$, for example, represents the interaction of U_d and V_r. The direct component of direct × residual interactions will always be specified first. We can therefore replace $U_d U_r$ by UU, $U_d V_r$ by UV, etc., without ambiguity.

5.5.1a A design criterion.

Now the components U_d, V_d, W_d of direct effects are estimated with full efficiency in all R-ortho designs. (See Theorems 3.1, 3.4 and 3.6). Also the efficiency in the estimation of any of the components U_r, V_r, W_r of residual effects is invariant over this set of designs. (See Theorem 3.7). The four components UU, UV, VU, VV of direct × residual interaction are likely to be more important than the other five components UW, VW, WU, WV, WW. A criterion for a suitable design is, therefore, that the components UU, UV, VU, VV are estimated efficiently and are uncorrelated with one another.

5.5.1b A model for 2 × 2 treatments.

The model can be expressed as

$$\begin{aligned} E\, y_1 &= \mu_1 1_{(n)} + \frac{1}{t}\left(X_1 \otimes 1^T_{(t)}\right) F_0^T (F_0 \alpha) + \varepsilon \\ E\, y_i &= \mu_i 1_{(n)} + X_{i,i-1} F_1^T (F_1 \alpha) + \varepsilon \qquad i = 2, \ldots, 8 \end{aligned} \tag{5.16}$$

where F_1 is a 10×16 matrix of normalised contrasts of U, V and W direct effects, U, V and W residual effects, and UU, UV, VU and VV direct $\times$ residual interaction. All direct $\times$ residual interaction components involving W_d or W_r are omitted. F_1 is given by

$$F_1 = \frac{1}{4} \times \begin{bmatrix} -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \end{bmatrix}$$

F_0 is a 10×16 matrix with the elements in the first three rows same as F_1 and zeros in the remaining seven rows. $F_1 F_1^T = I_{(10)}$ and $F_0 F_0^T = \begin{pmatrix} I_{(3)} & \underset{(3\times7)}{0} \\ \underset{(7\times3)}{0} & \underset{(7\times7)}{0} \end{pmatrix}$ where $\underset{(i\times j)}{0}$ is an $(i \times j)$ matrix with all elements equal to zero. $F_1^T F_1$ and $F_0^T F_0$ are idempotent matrices. The other terms are already defined in Chapter Three.

The normal equations are

$$\begin{aligned} F_0 X_{10}^T y_1 + F_1 \sum_{i=2}^{8} X_{i,i-1}^T y_i = {} & F_0 X_{10}^T 1_{(n)} \mu_1 + Q_1 \sum_{i=2} X_{i,i-1} \mu_i 1_{(n)} \\ & + F_0 X_{10}^T X_{10} F_0^T (F_0 \hat{\alpha}) + F_1 \sum_{i=2}^{8} X_{i,i-1}^T X_{i,i-1} F_1^T \cdot (F_1 \hat{\alpha}) \\ & + (F_0 N_0 + F_1 N) \end{aligned} \qquad (5.17)$$

$$\sum_{i=1}^{8} y_i = \sum_{i=1} \mu_i 1_{(n)} + N_0^T F_0^T \cdot (F_0 \hat{\alpha}) + N^T F_1^T \cdot (F_1 \hat{\alpha}) + 8\hat{\delta}$$

where $N_0 = X_{10}^T$, $N = \sum_{i=2}^{8} X_{i,i-1}^T$.

The information matrix eliminating subjects is

$$C_F = F_0 X_{10}^T X_{10} F_0^T + F_1 \sum_{i=2} X_{i,i-1}^T X_{i,i-1} F_1^T$$
$$- \frac{1}{8} (F_0 N_0 + F_1 N)(N_0^T F_0^T + N^T F_1^T) \ . \qquad (5.18)$$

The variance-covariance matrix is then

$$V_F = C_F^{-1} \sigma^2 \quad .$$

Let f_{ij} be the element in cell (i, j) of C_F^{-1}. Then the efficiency factors in the estimation of UU, UV, VU and VV direct × residual interaction are

$$EUU_{16} = \frac{1}{7 f_{77}} \quad ,$$

$$EUV_{16} = \frac{1}{7 f_{88}} \quad ,$$

$$EVU_{16} = \frac{1}{7 f_{99}} \quad ,$$

$$EVV_{16} = \frac{1}{7 f_{10,10}} \quad .$$

5.5.1c Designs for 2 × 2 treatments.

In this subsection, we will consider the suitability of designs by Berenblut (1964) and Patterson (1970) and the System A designs constructed by Partitions A1, A2 and A3. We will clearly prefer designs with diagonal V_F in which the elements f_{ii} are as small as possible.

All System A designs (including designs by Berenblut (1964) and Patterson (1970)) are R-orthogonal so that the submatrix V_{FS}, consisting of the first six rows and six columns of V_F, is diagonal. Also the values of f_{ii}, $i = 1, 2, \ldots, 6$ are invariant (that is, the same for each R-ortho design). In fact,

$$f_{11} = f_{22} = f_{33} = 1/8 \ ,$$
$$f_{44} = f_{55} = f_{66} = 8/55 \ .$$

In contrast, the values of f_{ii} , $i = 7, 8, \ldots, 10$ are not invariant. Hence EUU_{16}, EUV_{16}, EVU_{16} and EVV_{16} vary from one design to another. Our aim is to choose designs for which these efficiency factors are as large as possible. For all the designs listed in Appendix 5.5, the matrix V_F is diagonal, the four efficiency factors are larger than 0.80 and their average is 0.946.

The designs in Appendix 5.5 fall into four subsets, each design in a subset having the same efficiency factors. In two of the subsets EUV_{16}, EVU_{16} and either EUU_{16} or EVV_{16} (but not both) are 0.982, that is, equal in value to the efficiency factor for residual effects. The efficiency factor for the fourth interaction component, however, is as low as 0.839.

In the other two small subsets, each with only two designs, EUV_{16} and EVU_{16} are reduced to 0.946: in one subset, EUU_{16} is 0.982 and EVV_{16} is 0.911; in the other subset, EUU_{16} is 0.911 and EVV_{16} is 0.982. These two small subsets appear to be preferable when all four interaction components are of equal interest.

It is of interest to consider whether designs that are suitable for 2×2 treatments are also useful when treatments are equally spaced levels of a single quantitative factor or are levels of a qualitative factor. The two designs with $EUU_{16} = 0.982$ and $EVV_{16} = 0.911$ both have efficiency factor $ELL_{16} = 0.968$ (ELL_{16} is defined in subsection 5.3.1b); the two designs with $EUU_{16} = 0.911$ and $EVV_{16} = 0.982$ both have $ELL_{16} = 0.925$. Hence the former two designs are better. Although not among the best designs for a quantitative factor given in Appendices 5.1 and 5.2, they are reasonably suitable for that application.

The designs with $EUU_{16} = 0.839$ and $EVV_{16} = 0.982$ are not suitable for a quantitative factor since the efficiency factor ELL_{16}

of all these designs is 0.891 . But the designs with $EUU_{16} = 0.982$ and $EVV_{16} = 0.839$ are reasonably suitable for a quantitative factor since ELL_{16} of these designs is 0.976 . A design in the latter set is in Table 4.1 . The design in Table 4.1 together with designs DD (1324)(1423) and DD (1423)(1423) are also suitable when arranged in blocks of four subjects for a quantitative factor.

On the other hand, the designs in Appendix 5.1 that are best for a quantitative factor are not useful for 2×2 treatments.

None of the designs in Appendix 5.5 is suitable for qualitative treatments. Similarly, the designs in Appendix 5.4 that are suitable for qualitative treatments are not useful for 2×2 treatments. (Compare Appendix 5.4 and 5.5).

5.5.2a Another design criterion.

Let the four components UU, UV, VU, VV of direct $\times$ residual interaction constitute Set I ; the other five components UW, VW, WU, WV, WW constitute Set II of direct $\times$ residual interaction.

Even though the actual estimation of Set II components may not be important, designs with Set I uncorrelated with Set II are to be preferred since this property ensures that estimates of Set I components are not biased by real Set II components.

5.5.2b Designs.

In this subsection, we consider the selection of designs with the components UU, UV, VU, VV in Set I estimated efficiency, and with Set I uncorrelated with Set II . Therefore, we will only examine designs in Appendix 5.5 for their suitability. For this purpose, we require 15×16 matrices F_0^* and F_1^* , the first ten rows of which are given by F_0 and F_1 . The remaining five rows of F_0^* have all elements zero ; the remaining five rows of F_1^* are the coefficients of the (normalised) components in Set II of direct $\times$ residual interaction.

They are

$$\frac{1}{4} \times \begin{bmatrix} -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \end{bmatrix} .$$

The matrices F_0 and F_1 in (5.16) are then replaced by F_0^* and F_1^* .

The information matrix eliminating subjects will be represented by C_F^* and the variance-covariance matrix by V_F^* where $V_F^* = C_F^{*-1} \sigma^2$; also let g_{ij} be the element in cell (i, j) of C_F^{*-1} . Then, for all R-ortho designs, Set I of direct × residual interaction is uncorrelated with Set II if

$$g_{ij} = 0 \text{ for all combinations of } i = 11, \dots, 15, \ j = 7, \dots, 10 .$$

The six designs of basic design class DD in Appendix 5.5 have Set I uncorrelated with Set II. In fact, V_F^* is a diagonal matrix for these designs, that is, all nine components of direct × residual interaction are uncorrelated. These designs are marked with an asterisk (*) in Appendix 5.5 . These six designs are also binary.

There are other designs not in Appendix 5.5 but have diagonal V_F^* . The other six binary designs in basic design class DD also have this property, that is, all twelve binary designs in basic design class DD have diagonal V_F^* . However, these other six binary designs do not estimate components UU, UV, VU, VV efficiently.

5.6 Recommendations.

From the results obtained in main sections 5.3, 5.4 and 5.5, we find that different designs are best for different types of treatments. In this section, we list what we consider to be the best designs for each type.

5.6.1a Three quantitative treatments.

In the case of $t = 3$ treatments, the sixteen designs of System A in Table 4.10 are all equally suitable. The value of efficiency factor ELL_9 is 0.867 in all these designs. However, the two binary designs constructed by Berenblut (1964) represented by $A_1(132)\ R_1(123)\ A_2(132)\ R_2(231)\ A_3(132)\ R_3(312)$ and $A_1(123)\ R_1(132)\ A_2(123)\ R_2(213)\ A_3(123)\ R_3(321)$ have an additional property in that the interaction component LL is uncorrelated with QL. Estimates of LL are not biased by real QL components in these two designs. For this reason we recommend the two binary designs by Berenblut (1964).

None of the 16 designs is suitable for arrangement in blocks of three subjects because of the serious loss of information on LL component.

5.6.1b Four quantitative treatments.

The design in Table 4.13 together with two designs represented by

A1 : $(DD)_1(1423)(1342)\ (DD)_2(1234)(1342)$,

A1 : $(DD)_1(1423)(1243)\ (DD)_2(1234)(1342)$

are recommended for blocks of 8 or 16 subjects. All three designs have the same $ELL_8 = ELL_{16} = 0.991$. These designs in blocks of 16 subjects also have the interaction component LL uncorrelated with QL.

The above three designs are not, however, suitable for blocks of four subjects. The best designs in blocks of four subjects are the following

DD(1423)(1324)

DD(1324)(1324)

DD(1324)(1423)

DD(1423)(1423)

C4 : $(DD)_1(1423)(1324)\ (DD)_2(1324)(1324)$

C4 : $(DD)_1(1324)(1324)\ (DD)_2(1423)(1324)$

C4 : $(DD)_1(1324)(1423)\ (DD)_2(1423)(1423)$

C4 : $(DD)_1(1423)(1423)\ (DD)_2(1324)(1423)$

These designs have the same $ELL_4 = 0.943$. Design DD(1423)(1324) is given in full in Table 4.1.

5.6.2a Three qualitative treatments.

For three qualitative treatments we recommend the same two designs as for three quantitative treatments. (See section 5.6.1a).

5.6.2b Four qualitative treatments.

For this case we recommend the six designs, in blocks of 16 subjects, that are D-, E- and A-optimum (see subsection 5.4.1b). These designs are represented by

AA(1432)(1342) , AA(1342)(1432) ,
BB(1243)(1423) , BB(1423)(1243) ,
CC(1324)(1234) , CC(1234)(1324) .

The six designs are constructed by Berenblut (1964).

5.6.3 2×2 treatments.

Four designs appear to be more suitable than the other for this case. These designs are

$$\begin{aligned}
A1 &: (CA)_1(1432)(1324)\ (CA)_2(1423)(1324)\ ,\\
A1 &: (CA)_1(1432)(1423)\ (CA)_2(1423)(1423)\ ,\\
A1 &: (CB)_1(1423)(1234)\ (CB)_2(1432)(1234)\ ,\\
A1 &: (CB)_1(1423)(1432)\ (CB)_2(1432)(1432)\ .
\end{aligned}$$

All four designs have $EUV_{16} = EVU_{16} = 0.946$. The first two designs have $EUU_{16} = 0.982$ and $EVV_{16} = 0.911$. The third and fourth designs have $EUU_{16} = 0.911$ and $EVV_{16} = 0.982$. There are many other designs with the average of the four efficiency factors the same as the above four designs. But the above four designs are more suitable when all four interaction components UU, UV, VU and VV are of equal interest.

Appendix 5.1 (see subsection 5.3.1d). Designs for four treatments with efficiency factor $ELL_{16} > 0.985$.

Designs with $ELL_{16} = 0.991$

+* A1 : $(DD)_1(1423)(1342)(DD)_2(1234)(1342)$

+* A1 : $(DD)_1(1432)(1324)(DD)_2(1324)(1324)$

+* A1 : $(DD)_1(1423)(1243)(DD)_2(1234)(1342)$

Designs with $ELL_{16} = 0.988$

* A2 : $(DA)_1(1324)(1423)(DA)_2(1432)(1423)$

* A2 : $(DA)_1(1432)(1324)(DA)_2(1324)(1324)$

+* A1 : $(DD)_1(1324)(1423)(CC)_2(1342)(1432)$

+* A1 : $(DD)_1(1423)(1423)(CC)_2(1243)(1432)$

+* A1 : $(DD)_1(1324)(1423)(CC)_2(1243)(1432)$

+* A1 : $(DD)_1(1423)(1423)(CC)_2(1342)(1432)$

A1 : $(DD)_1(1432)(1324)(AD)_2(1423)(1324)$

A1 : $(DD)_1(1432)(1324)(AD)_2(1324)(1324)$

A1 : $(AD)_1(1423)(1423)(DD)_2(1342)(1423)$

A1 : $(AD)_1(1324)(1423)(DD)_2(1243)(1423)$

A1 : $(AD)_1(1324)(1423)(DD)_2(1342)(1423)$

A1 : $(AD)_1(1423)(1423)(DD)_2(1243)(1423)$

A1 : $(AD)_1(1324)(1423)(CC)_2(1243)(1432)$

A1 : $(AD)_1(1423)(1423)(CC)_2(1342)(1432)$

A1 : $(AD)_1(1423)(1423)(CC)_2(1243)(1432)$

A1 : $(AD)_1(1324)(1423)(CC)_2(1342)(1432)$

+* A1 : $(DD)_1(1324)(1423)(DD)_2(1342)(1423)$

or C5 : $(BD)_1(1324)(1423)(BD)_2(1342)(1423)$

+* A1 : $(DD)_1(1423)(1423)(DD)_2(1243)(1423)$

or C5 : $(BD)_1(1423)(1423)(BD)_2(1243)(1423)$

Appendix 5.1 (continued)

+* A1 : $(DD)_1(1324)(1423)(DD)_2(1243)(1423)$

+* A1 : $(DD)_1(1423)(1423)(DD)_2(1342)(1423)$

Designs with $ELL_{16} = 0.986$

A2 : $(AA)_1(1423)(1423)(DA)_2(1432)(1423)$

A2 : $(AA)_1(1324)(1423)(DA)_2(1432)(1423)$

A2 : $(DA)_1(1432)(1324)(AA)_2(1423)(1324)$

A2 : $(DA)_1(1432)(1324)(AA)_2(1324)(1324)$

+* A1 : $(DD)_1(1423)(1243)(CC)_2(1234)(1243)$

+* A1 : $(DD)_1(1423)(1243)(CC)_2(1324)(1243)$

+* A1 : $(DD)_1(1423)(1342)(CC)_2(1234)(1243)$

+* A1 : $(DD)_1(1432)(1324)(CC)_2(1234)(1234)$

+* A1 : $(DD)_1(1432)(1324)(CC)_2(1324)(1234)$

+* A1 : $(DD)_1(1423)(1342)(CC)_2(1324)(1243)$

+* A1 : $(CC)_1(1432)(1324)(DD)_2(1324)(1324)$ b

+* A1 : $(CC)_1(1432)(1342)(DD)_2(1234)(1342)$

+* A1 : $(CC)_1(1423)(1342)(DD)_2(1234)(1342)$

+* A1 : $(CC)_1(1423)(1324)(DD)_2(1324)(1324)$

+* A1 : $(CC)_1(1432)(1243)(DD)_2(1234)(1342)$

+* A1 : $(CC)_1(1423)(1243)(DD)_2(1234)(1342)$

A1 : $(DD)_1(1324)(1423)(AD)_2(1234)(1423)$

A1 : $(DD)_1(1423)(1423)(AD)_2(1432)(1423)$

A1 : $(DD)_1(1324)(1423)(AD)_2(1432)(1423)$

A1 : $(DD)_1(1423)(1423)(AD)_2(1234)(1423)$

A1 : $(DD)_1(1324)(1423)(AD)_2(1342)(1423)$

A1 : $(DD)_1(1423)(1423)(AD)_2(1243)(1423)$

Appendix 5.1 (continued)

A1 : $(DD)_1(1324)(1423)(AD)_2(1243)(1423)$

A1 : $(DD)_1(1423)(1423)(AD)_2(1342)(1423)$

A1 : $(AD)_1(1324)(1423)(AD)_2(1234)(1423)$

A1 : $(AD)_1(1423)(1423)(AD)_2(1432)(1423)$

A1 : $(AD)_1(1324)(1423)(AD)_2(1432)(1423)$

A1 : $(AD)_1(1423)(1423)(AD)_2(1234)(1423)$

A1 : $(AD)_1(1324)(1423)(AD)_2(1342)(1423)$

A1 : $(AD)_1(1423)(1423)(AD)_2(1243)(1423)$

A1 : $(AD)_1(1324)(1423)(AD)_2(1243)(1423)$

A1 : $(AD)_1(1423)(1423)(AD)_2(1342)(1423)$

All designs marked * can be arranged in blocks of 8 with $ELL_8 = ELL_{16}$ and no loss of information on direct and residual effects. All designs marked + have interaction component LL uncorrelated with QL. The design marked with b is the only binary design in Appendix 5.1. We observe in Appendix 5.1 (and also in Appendix 5.2) that there are other designs, besides those of Berenblut (1964) and Patterson (1970), that belong to more than one system of SF designs.

Appendix 5.2 (see subsection 5.3.1d). Designs for four treatments with efficiency factor ELL_{16} in the range 0.980 - 0.985 (inclusive)

Designs with $ELL_{16} = 0.985$

A2 : $(AA)_1(1342)(1324)(DA)_2(1324)(1324)$

A2 : $(AA)_1(1432)(1324)(DA)_2(1324)(1324)$

A2 : $(DA)_1(1324)(1423)(AA)_2(1342)(1423)$

A2 : $(DA)_1(1324)(1423)(AA)_2(1432)(1423)$

Appendix 5.2 (continued)

* A2 : $(DA)_1(1432)(1324)(DA)_2(1423)(1324)$

or C2 : $(CA)_1(1432)(1324)(CA)_2(1423)(1324)$

* A2 : $(DA)_1(1423)(1423)(DA)_2(1432)(1423)$

or C2 : $(CA)_1(1423)(1423)(CA)_2(1432)(1423)$

A1 : $(DD)_1(1324)(1423)(BD)_2(1234)(1423)$

A1 : $(DD)_1(1423)(1423)(BD)_2(1432)(1423)$

A1 : $(DD)_1(1324)(1423)(BD)_2(1432)(1423)$

A1 : $(DD)_1(1423)(1423)(BD)_2(1234)(1423)$

A1 : $(AD)_1(1324)(1423)(DD)_2(1234)(1423)$

A1 : $(AD)_1(1423)(1423)(DD)_2(1432)(1423)$

A1 : $(AD)_1(1423)(1423)(DD)_2(1234)(1423)$

A1 : $(AD)_1(1324)(1423)(DD)_2(1432)(1423)$

A1 : $(AD)_1(1324)(1423)(BD)_2(1234)(1423)$

A1 : $(AD)_1(1423)(1423)(BD)_2(1432)(1423)$

A1 : $(AD)_1(1423)(1423)(BD)_2(1234)(1423)$

A1 : $(AD)_1(1324)(1423)(BD)_2(1432)(1423)$

+* A1 : $(DD)_1(1432)(1324)(DD)_2(1423)(1324)$

or C1 : $(CD)_1(1432)(1324)(CD)_2(1423)(1324)$

+* A1 : $(DD)_1(1324)(1423)(DD)_2(1234)(1423)$

or C1 : $(CD)_1(1324)(1423)(CD)_2(1234)(1423)$

+* A1 : $(DD)_1(1423)(1423)(DD)_2(1432)(1423)$

or C1 : $(CD)_1(1423)(1423)(CD)_2(1432)(1423)$

+* A1 : $(DD)_1(1324)(1423)(DD)_2(1432)(1423)$

+* A1 : $(DD)_1(1423)(1423)(DD)_2(1234)(1423)$

Designs with $ELL_{16} = 0.984$

* A2 : $(DA)_1(1324)(1423)(CA)_2(1432)(1423)$

* A2 : $(DA)_1(1324)(1423)(CA)_2(1423)(1423)$

Appendix 5.2 (continued)

* A2 : $(CA)_1(1432)(1324)(DA)_2(1324)(1324)$

* A2 : $(CA)_1(1423)(1324)(DA)_2(1324)(1324)$

\+ A2 : $(AA)_1(1432)(1324)(AA)_2(1423)(1324)$

A2 : $(AA)_1(1423)(1423)(AA)_2(1432)(1423)$

A2 : $(AA)_1(1342)(1324)(AA)_2(1423)(1324)$

A2 : $(AA)_1(1324)(1423)(AA)_2(1432)(1423)$

A2 : $(AA)_1(1342)(1324)(AA)_2(1324)(1324)$

\+ A2 : $(AA)_1(1324)(1423)(AA)_2(1342)(1423)$

A2 : $(AA)_1(1432)(1324)(AA)_2(1324)(1324)$

A2 : $(AA)_1(1423)(1423)(AA)_2(1342)(1423)$

A1 : $(AD)_1(1432)(1324)(DD)_2(1324)(1324)$

A1 : $(AD)_1(1342)(1324)(DD)_2(1324)(1324)$

A1 : $(CC)_1(1423)(1324)(AD)_2(1423)(1324)$

A1 : $(CC)_1(1432)(1324)(AD)_2(1423)(1324)$

A1 : $(CC)_1(1432)(1324)(AD)_2(1324)(1324)$

A1 : $(CC)_1(1423)(1324)(AD)_2(1324)(1324)$

Designs with $ELL_{16} = 0.982$

A2 : $(AA)_1(1432)(1324)(DA)_2(1423)(1324)$

A2 : $(AA)_1(1342)(1324)(DA)_2(1423)(1324)$

A2 : $(DA)_1(1423)(1423)(AA)_2(1432)(1423)$

A2 : $(DA)_1(1423)(1423)(AA)_2(1342)(1423)$

A2 : $(AA)_1(1423)(1423)(CA)_2(1423)(1423)$

A2 : $(AA)_1(1423)(1423)(CA)_2(1432)(1423)$

A2 : $(AA)_1(1324)(1423)(CA)_2(1432)(1423)$

A2 : $(AA)_1(1324)(1423)(CA)_2(1423)(1423)$

A2 : $(CA)_1(1423)(1324)(AA)_2(1423)(1324)$

A2 : $(CA)_1(1432)(1324)(AA)_2(1423)(1324)$

Appendix 5.2 (continued)

A2 : $(CA)_1(1432)(1324)(AA)_2(1324)(1324)$

A2 : $(CA)_1(1423)(1324)(AA)_2(1324)(1324)$

* A2 : $(DA)_1(1342)(1324)(DA)_2(1324)(1324)$

or C6 : $(BA)_1(1342)(1324)(BA)_2(1324)(1324)$

* A2 : $(DA)_1(1324)(1423)(DA)_2(1342)(1423)$

or C6 : $(BA)_1(1324)(1423)(BA)_2(1342)(1423)$

* A1 : $(DD)_1(1324)(1423)(BD)_2(1324)(1423)$

or C5 : $(BD)_1(1324)(1423)(DD)_2(1324)(1423)$

* A1 : $(DD)_1(1423)(1423)(BD)_2(1423)(1423)$

or C5 : $(BD)_1(1423)(1423)(DD)_2(1423)(1423)$

* A1 : $(DD)_1(1324)(1423)(BD)_2(1342)(1423)$

or C5 : $(DD)_1(1324)(1423)(BD)_2(1342)(1423)$

* A1 : $(DD)_1(1423)(1423)(BD)_2(1243)(1423)$

or C5 : $(DD)_1(1423)(1423)(BD)_2(1243)(1423)$

A1 : $(DD)_1(1324)(1423)(BD)_2(1243)(1423)$

A1 : $(DD)_1(1423)(1423)(BD)_2(1342)(1423)$

A1 : $(DD)_1(1324)(1423)(BD)_2(1423)(1423)$

A1 : $(DD)_1(1423)(1423)(BD)_2(1324)(1423)$

A1 : $(AD)_1(1324)(1423)(BD)_2(1324)(1423)$

A1 : $(AD)_1(1423)(1423)(BD)_2(1423)(1423)$

A1 : $(AD)_1(1423)(1423)(BD)_2(1342)(1423)$

A1 : $(AD)_1(1324)(1423)(BD)_2(1243)(1423)$

A1 : $(AD)_1(1324)(1423)(BD)_2(1342)(1423)$

A1 : $(AD)_1(1423)(1423)(BD)_2(1243)(1423)$

A1 : $(AD)_1(1423)(1423)(BD)_2(1324)(1423)$

A1 : $(AD)_1(1324)(1423)(BD)_2(1423)(1423)$

\+ * A1 : $(DD)_1(1432)(1324)(DD)_2(1234)(1324)$

or C5 : $(BD)_1(1432)(1324)(BD)_2(1234)(1324)$

Appendix 5.2 (continued)

+ * A1 : $(DD)_1(1432)(1342)(DD)_2(1234)(1342)$
or C5 : $(BD)_1(1432)(1342)(BD)_2(1234)(1342)$
+ * A1 : $(DD)_1(1423)(1324)(DD)_2(1324)(1324)$
or C4 : $(AD)_1(1423)(1324)(AD)_2(1324)(1324)$
+ * A1 : $(DD)_1(1423)(1342)(DD)_2(1324)(1342)$
or C4 : $(AD)_1(1423)(1342)(AD)_2(1324)(1342)$
+ * A1 : $(DD)_1(1432)(1243)(DD)_2(1234)(1342)$
or C5 : $(BC)_1(1432)(1243)(BC)_2(1234)(1243)$
+ * A1 : $(DD)_1(1423)(1243)(DD)_2(1324)(1342)$
or C4 : $(AC)_1(1423)(1243)(AC)_2(1324)(1243)$
+ * A1 : $(CC)_1(1423)(1243)(CC)_2(1234)(1243)$
+ * A1 : $(CC)_1(1432)(1243)(CC)_2(1234)(1243)$
+ * A1 : $(CC)_1(1432)(1243)(CC)_2(1324)(1243)$
+ * A1 : $(CC)_1(1423)(1324)(CC)_2(1234)(1234)$
+ * A1 : $(CC)_1(1423)(1342)(CC)_2(1234)(1243)$
+ * A1 : $(CC)_1(1432)(1324)(CC)_2(1234)(1234)$
+ * A1 : $(CC)_1(1432)(1342)(CC)_2(1234)(1243)$
+ * A1 : $(CC)_1(1432)(1324)(CC)_2(1324)(1234)$
+ * A1 : $(CC)_1(1432)(1342)(CC)_2(1324)(1243)$
+ * A1 : $(CC)_1(1423)(1324)(CC)_2(1324)(1234)$
+ * A1 : $(CC)_1(1423)(1342)(CC)_2(1324)(1243)$
* A1 : $(DB)_1(1423)(1324)(CB)_2(1234)(1324)$
* A1 : $(DB)_1(1423)(1342)(CB)_2(1234)(1342)$
* A1 : $(DB)_1(1432)(1324)(CB)_2(1234)(1324)$
* A1 : $(DB)_1(1432)(1342)(CB)_2(1234)(1342)$
* A1 : $(DB)_1(1432)(1324)(CB)_2(1324)(1324)$
* A1 : $(DB)_1(1432)(1342)(CB)_2(1324)(1342)$
+ * A1 : $(CC)_1(1423)(1243)(CC)_2(1324)(1243)$

Appendix 5.2 (continued)

* A1 : $(DB)_1(1423)(1324)(CB)_2(1324)(1324)$

or C4 : $(AB)_1(1423)(1324)(AB)_2(1324)(1324)$

* A1 : $(DB)_1(1423)(1342)(CB)_2(1324)(1342)$

or C4 : $(AB)_1(1423)(1342)(AB)_2(1324)(1342)$

* A1 : $(CB)_1(1432)(1324)(DB)_2(1324)(1324)$

* A1 : $(CB)_1(1432)(1342)(DB)_2(1324)(1342)$

* A1 : $(CB)_1(1432)(1324)(DB)_2(1234)(1324)$

* A1 : $(CB)_1(1432)(1342)(DB)_2(1234)(1342)$

* A1 : $(CB)_1(1423)(1324)(DB)_2(1234)(1324)$

* A1 : $(CB)_1(1423)(1342)(DB)_2(1234)(1342)$

* A1 : $(CB)_1(1423)(1324)(DB)_2(1324)(1324)$

* A1 : $(CB)_1(1423)(1342)(DB)_2(1324)(1342)$

* A1 : $(DB)_1(1423)(1324)(DB)_2(1234)(1324)$

* A1 : $(DB)_1(1423)(1342)(DB)_2(1234)(1342)$

\+ * A1 : $(DB)_1(1432)(1324)(DB)_2(1234)(1324)$

or C5 : $(BB)_1(1432)(1324)(BB)_2(1234)(1324)$

\+ * A1 : $(DB)_1(1432)(1342)(DB)_2(1234)(1342)$

or C5 : $(BB)_1(1432)(1342)(BB)_2(1234)(1342)$

* A1 : $(DB)_1(1432)(1324)(DB)_2(1324)(1324)$

* A1 : $(DB)_1(1432)(1342)(DB)_2(1324)(1342)$

* A1 : $(DB)_1(1423)(1324)(DB)_2(1324)(1324)$

* A1 : $(DB)_1(1423)(1342)(DB)_2(1324)(1342)$

* A1 : $(CB)_1(1423)(1324)(CB)_2(1234)(1324)$

* A1 : $(CB)_1(1423)(1342)(CB)_2(1234)(1342)$

* A1 : $(CB)_1(1432)(1324)(CB)_2(1234)(1324)$

* A1 : $(CB)_1(1432)(1342)(CB)_2(1234)(1342)$

* A1 : $(CB)_1(1432)(1324)(CB)_2(1324)(1324)$

Appendix 5.2 (continued)

* A1 : $(CB)_1(1432)(1342)(CB)_2(1324)(1342)$

* A1 : $(CB)_1(1423)(1324)(CB)_2(1324)(1324)$

* A1 : $(CB)_1(1423)(1342)(CB)_2(1324)(1342)$

+* B1 : $(DD)_1(1342)(1234)(DD)_2(1342)(1432)$

or C2 : $(DB)_1(1342)(1234)(DB)_2(1342)(1432)$

+* B1 : $(DD)_1(1342)(1324)(DD)_2(1342)(1423)$

or C3 : $(DA)_1(1342)(1324)(DA)_2(1342)(1423)$

+* B1 : $(DD)_1(1342)(1234)(DD)_2(1243)(1432)$

or C2 : $(CB)_1(1342)(1234)(CB)_2(1342)(1432)$

+* B1 : $(DD)_1(1342)(1324)(DD)_2(1243)(1423)$

or C3 : $(CA)_1(1342)(1324)(CA)_2(1342)(1423)$

+* B1 : $(CC)_1(1342)(1324)(CC)_2(1342)(1432)$

+* B1 : $(CC)_1(1342)(1234)(CC)_2(1342)(1423)$

+* B1 : $(CC)_1(1342)(1324)(CC)_2(1243)(1432)$

+* B1 : $(CC)_1(1342)(1234)(CC)_2(1243)(1423)$

* C2 : $(DB)_1(1342)(1234)(CB)_2(1342)(1432)$

* C2 : $(CB)_1(1342)(1234)(DB)_2(1342)(1432)$

* C3 : $(DA)_1(1342)(1324)(CA)_2(1342)(1423)$

* C3 : $(CA)_1(1342)(1324)(DA)_2(1342)(1423)$

* C4 : $(AD)_1(1423)(1342)(AC)_2(1324)(1243)$

* C4 : $(AC)_1(1423)(1243)(AD)_2(1324)(1342)$

* C5 : $(BD)_1(1432)(1342)(BC)_2(1234)(1243)$

* C5 : $(BC)_1(1432)(1243)(BD)_2(1234)(1342)$

Designs with $ELL_{16} = 0.981$

A2 : $(AA)_1(1324)(1423)(DA)_2(1342)(1423)$

A2 : $(AA)_1(1423)(1423)(DA)_2(1342)(1423)$

A2 : $(DA)_1(1342)(1324)(AA)_2(1324)(1324)$

Appendix 5.2 (continued)

A2 : $(DA)_1(1342)(1324)(AA)_2(1423)(1324)$

* A2 : $(DA)_1(1423)(1423)(CA)_2(1423)(1423)$

or C2 : $(CA)_1(1423)(1423)(DA)_2(1423)(1423)$

* A2 : $(DA)_1(1423)(1423)(CA)_2(1432)(1423)$

or C2 : $(DA)_1(1423)(1423)(CA)_2(1432)(1423)$

* A2 : $(CA)_1(1423)(1324)(DA)_2(1423)(1324)$

or C2 : $(DA)_1(1423)(1324)(CA)_2(1423)(1324)$

* A2 : $(CA)_1(1432)(1324)(DA)_2(1423)(1324)$

or C2 : $(CA)_1(1432)(1324)(DA)_2(1423)(1324)$

+* A1 : $(DD)_1(1324)(1423)(CC)_2(1324)(1432)$

or C1 : $(CD)_1(1324)(1423)(DD)_2(1324)(1423)$

+* A1 : $(DD)_1(1423)(1423)(CC)_2(1423)(1432)$

or C1 : $(CD)_1(1423)(1423)(DD)_2(1423)(1423)$

+* A1 : $(DD)_1(1324)(1423)(CC)_2(1234)(1432)$

or C1 : $(DD)_1(1324)(1423)(CD)_2(1234)(1423)$

+* A1 : $(DD)_1(1423)(1423)(CC)_2(1432)(1432)$

or C1 : $(DD)_1(1423)(1423)(CD)_2(1432)(1423)$

+* A1 : $(DD)_1(1324)(1423)(CC)_2(1432)(1432)$

+* A1 : $(DD)_1(1423)(1423)(CC)_2(1234)(1432)$

+* A1 : $(DD)_1(1324)(1423)(CC)_2(1423)(1432)$

+* A1 : $(DD)_1(1423)(1423)(CC)_2(1324)(1423)$

+* A1 : $(CC)_1(1423)(1324)(DD)_2(1423)(1324)$

or C1 : $(DD)_1(1423)(1324)(CD)_2(1423)(1324)$

+* A1 : $(CC)_1(1432)(1324)(DD)_2(1423)(1324)$

or C1 : $(CD)_1(1432)(1324)(DD)_2(1423)(1324)$

A1 : $(AD)_1(1423)(1423)(CC)_2(1423)(1432)$

A1 : $(AD)_1(1324)(1423)(CC)_2(1324)(1432)$

A1 : $(AD)_1(1423)(1423)(CC)_2(1432)(1432)$

A1 : $(AD)_1(1324)(1423)(CC)_2(1234)(1432)$

A1 : $(AD)_1(1324)(1423)(CC)_2(1432)(1432)$

Appendix 5.2 (continued)

A1 : $(AD)_1(1423)(1423)(CC)_2(1234)(1432)$

A1 : $(AD)_1(1324)(1423)(CC)_2(1423)(1432)$

A1 : $(AD)_1(1423)(1423)(CC)_2(1324)(1432)$

A1 : $(AD)_1(1432)(1324)(AD)_2(1423)(1324)$

A1 : $(AD)_1(1342)(1324)(AD)_2(1423)(1324)$

A1 : $(AD)_1(1342)(1324)(AD)_2(1324)(1324)$

A1 : $(AD)_1(1432)(1324)(AD)_2(1324)(1324)$

All designs marked * can be arranged in blocks of 8 with $ELL_8 = ELL_{16}$ and no loss of information on direct and residual effects. All designs marked + have interaction component LL uncorrelated with QL.

Appendix 5.3 (see subsection 5.3.3d). Designs for four treatments in blocks of four subjects with efficiency factor $ELL_4 > 0.920$.

Designs with $ELL_4 = 0.943$

	ELL_{16}
DD(1423)(1324)	0.976
DD(1324)(1324)	0.965
DD(1324)(1423)	0.976
DD(1423)(1423)	0.976
C4 : $(DD)_1(1423)(1324)(DD)_2(1324)(1324)$	0.971
C4 : $(DD)_1(1324)(1324)(DD)_2(1423)(1324)$	0.971
C4 : $(DD)_1(1324)(1423)(DD)_2(1423)(1423)$	0.976
C4 : $(DD)_1(1423)(1423)(DD)_2(1324)(1423)$	0.976

Designs with $ELL_4 = 0.931$

C3 : $(DD)_1(1423)(1324)(DD)_2(1432)(1324)$	0.968
C3 : $(DD)_1(1432)(1324)(DD)_2(1423)(1324)$	0.968
C3 : $(DD)_1(1234)(1423)(DD)_2(1324)(1423)$	0.968

Appendix 5.3 (continued)

C3 : $(DD)_1(1324)(1423)(DD)_2(1234)(1423)$	0.968
C3 : $(DD)_1(1432)(1423)(DD)_2(1423)(1423)$	0.968
C3 : $(DD)_1(1423)(1423)(DD)_2(1432)(1423)$	0.968
C6 : $(DD)_1(1342)(1324)(DD)_2(1324)(1324)$	0.954
C6 : $(DD)_1(1324)(1324)(DD)_2(1342)(1324)$	0.954
C6 : $(DD)_1(1324)(1423)(DD)_2(1342)(1423)$	0.965
C6 : $(DD)_1(1342)(1423)(DD)_2(1324)(1423)$	0.965
C6 : $(DD)_1(1423)(1423)(DD)_2(1243)(1423)$	0.965
C6 : $(DD)_1(1243)(1423)(DD)_2(1423)(1423)$	0.965

Design with $ELL_4 = 0.920$

	ELL_{16}
DD(1423)(1234)	0.959
DD(1432)(1324)	0.959
DD(1342)(1324)	0.942
DD(1324)(1342)	0.942
DD(1423)(1342)	0.954
DD(1423)(1432)	0.959
DD(1324)(1432)	0.959
DD(1234)(1423)	0.959
DD(1342)(1423)	0.954
DD(1432)(1423)	0.959
DD(1243)(1423)	0.954
DD(1423)(1243)	0.954
C3 : $(DD)_1(1342)(1423)(DD)_2(1243)(1423)$	0.954
C3 : $(DD)_1(1243)(1423)(DD)_2(1342)(1423)$	0.954
C3 : $(DD)_1(1342)(1324)(DD)_2(1342)(1423)$	0.948
C3 : $(DD)_1(1342)(1423)(DD)_2(1342)(1324)$	0.948

Appendix 5.3 (continued)

C3 : $(DD)_1(1342)(1324)(DD)_2(1243)(1423)$	0.948
C3 : $(DD)_1(1243)(1423)(DD)_2(1342)(1324)$	0.948
C4 : $(DD)_1(1432)(1324)(DD)_2(1342)(1324)$	0.951
C4 : $(DD)_1(1342)(1324)(DD)_2(1432)(1324)$	0.951
C4 : $(DD)_1(1324)(1342)(DD)_2(1423)(1342)$	0.948
C4 : $(DD)_1(1423)(1342)(DD)_2(1324)(1342)$	0.948
C4 : $(DD)_1(1423)(1432)(DD)_2(1324)(1432)$	0.959
C4 : $(DD)_1(1324)(1432)(DD)_2(1423)(1432)$	0.959
C4 : $(DD)_1(1234)(1423)(DD)_2(1243)(1423)$	0.956
C4 : $(DD)_1(1342)(1423)(DD)_2(1432)(1423)$	0.956
C4 : $(DD)_1(1432)(1423)(DD)_2(1342)(1423)$	0.956
C4 : $(DD)_1(1243)(1423)(DD)_2(1234)(1423)$	0.956
C4 : $(DD)_1(1423)(1342)(DD)_2(1423)(1243)$	0.954
C4 : $(DD)_1(1423)(1243)(DD)_2(1423)(1342)$	0.954
C4 : $(DD)_1(1324)(1342)(DD)_2(1423)(1243)$	0.948
C4 : $(DD)_1(1423)(1243)(DD)_2(1324)(1342)$	0.948
C6 : $(DD)_1(1234)(1423)(DD)_2(1432)(1423)$	0.959
C6 : $(DD)_1(1432)(1423)(DD)_2(1234)(1423)$	0.959

Appendix 5.4 (see subsection 5.4.1d). Values of λ_{max}, $\sum_{i=1}^{9} \lambda_i$ and $\prod_{i=1}^{9} \lambda_i$ in designs for four qualitative treatments.

(Limit : $\sum_{i=1}^{9} \lambda_i < 1.46$, $\prod_{i=1}^{9} \lambda_i < 7.519 \times 10^{-8}$)

Similar designs with $\lambda_{max} = 0.1702$, $\sum_{i=1}^{9} \lambda_i = 1.4518$, $\prod_{i=1}^{9} \lambda_i = 7.347 \times 10^{-8}$

AA(1432)(1342) , BB(1243)(1423) , CC(1324)(1234)

AA(1342)(1432) , BB(1423)(1243) , CC(1234)(1324)

Appendix 5.4 (continued)

Similar designs with $\lambda_{max} = 0.1702$, $\sum_{i=1}^{9} \lambda_i = 1.4576$, $\prod_{i=1}^{9} \lambda_i = 7.484 \times 10^{-8}$.

AA(1432)(1234) , BB(1423)(1324) , CC(1423)(1234)

AA(1234)(1342) , BB(1243)(1342) , CC(1432)(1324)

AA(1243)(1432) , BB(1342)(1423) , CC(1234)(1432)

AA(1342)(1243) , BB(1324)(1243) , CC(1324)(1423)

Similar designs with $\lambda_{max} = 0.1808$, $\sum_{i=1}^{9} \lambda_i = 1.4575$, $\prod_{i=1}^{9} \lambda_i = 7.478 \times 10^{-8}$.

A3 : $(DD)_1$(1234)(1432)$(AA)_2$(1342)(1432)

A3 : $(DD)_1$(1243)(1342)$(AA)_2$(1432)(1342)

A3 : $(AA)_1$(1342)(1432)$(DD)_2$(1243)(1342)

A3 : $(AA)_1$(1432)(1342)$(DD)_2$(1234)(1432)

A2 : $(DD)_1$(1342)(1243)$(BB)_2$(1423)(1243)

A2 : $(DD)_1$(1324)(1423)$(BB)_2$(1243)(1423)

A2 : $(BB)_1$(1243)(1423)$(DD)_2$(1342)(1243)

A2 : $(BB)_1$(1423)(1243)$(DD)_2$(1324)(1423)

A1 : $(DD)_1$(1423)(1324)$(CC)_2$(1234)(1324)

A1 : $(DD)_1$(1432)(1234)$(CC)_2$(1324)(1234)

A1 : $(CC)_1$(1234)(1324)$(DD)_2$(1432)(1234)

A1 : $(CC)_1$(1324)(1234)$(DD)_2$(1423)(1324)

Similar designs with $\lambda_{max} = 0.1875$, $\sum_{i=1}^{9} \lambda_i = 1.4567$, $\prod_{i=1}^{9} \lambda_i = 7.458 \times 10^{-8}$.

A3 : $(DD)_1$(1342)(1342)$(AA)_2$(1432)(1342)

A3 : $(DD)_1$(1432)(1432)$(AA)_2$(1342)(1432)

A3 : $(AA)_1$(1342)(1432)$(DD)_2$(1342)(1342)

A3 : $(AA)_1$(1432)(1342)$(DD)_2$(1432)(1432)

A2 : $(DD)_1$(1423)(1423)$(BB)_2$(1243)(1423)

A2 : $(DD)_1$(1243)(1243)$(BB)_2$(1423)(1243)

A2 : $(BB)_1$(1423)(1243)$(DD)_2$(1423)(1423)

Appendix 5.4 (continued)

A2 : $(BB)_1(1243)(1423)(DD)_2(1243)(1243)$

A1 : $(DD)_1(1234)(1234)(CC)_2(1324)(1234)$

A1 : $(DD)_1(1324)(1324)(CC)_2(1234)(1324)$

A1 : $(CC)_1(1234)(1324)(DD)_2(1234)(1234)$

A1 : $(CC)_1(1324)(1234)(DD)_2(1324)(1324)$

Similar designs with $\lambda_{max} = 0.1778$, $\sum_{i=1}^{9} \lambda_i = 1.4578$, $\prod_{i=1}^{9} \lambda_i = 7.486 \times 10^{-8}$.

A3 : $(DD)_1(1432)(1234)(DD)_2(1342)(1243)$

A3 : $(DD)_1(1243)(1342)(DD)_2(1234)(1432)$

A3 : $(DD)_1(1234)(1432)(DD)_2(1243)(1342)$

A3 : $(DD)_1(1342)(1243)(DD)_2(1432)(1234)$

A2 : $(DD)_1(1423)(1324)(DD)_2(1243)(1342)$

A2 : $(DD)_1(1243)(1342)(DD)_2(1423)(1324)$

A2 : $(DD)_1(1324)(1423)(DD)_2(1342)(1243)$

A2 : $(DD)_1(1342)(1243)(DD)_2(1324)(1423)$

A1 : $(DD)_1(1432)(1234)(DD)_2(1423)(1324)$

A1 : $(DD)_1(1423)(1324)(DD)_2(1432)(1234)$

A1 : $(DD)_1(1234)(1432)(DD)_2(1324)(1423)$

A1 : $(DD)_1(1324)(1423)(DD)_2(1234)(1432)$

Similar designs with $\lambda_{max} = 0.1813$, $\sum_{i=1}^{9} \lambda_i = 1.4585$, $\prod_{i=1}^{9} \lambda_i = 7.502 \times 10^{-8}$.

A3 : $(AA)_1(1243)(1432)(AA)_2(1342)(1432)$

A3 : $(AA)_1(1342)(1432)(AA)_2(1243)(1432)$

A3 : $(AA)_1(1234)(1342)(AA)_2(1432)(1342)$

A3 : $(AA)_1(1432)(1342)(AA)_2(1234)(1342)$

A3 : $(AA)_1(1234)(1342)(AA)_2(1342)(1432)$

Appendix 5.4 (continued)

A3 : $(AA)_1(1432)(1342)(AA)_2(1243)(1432)$

A3 : $(AA)_1(1243)(1432)(AA)_2(1432)(1342)$

A3 : $(AA)_1(1342)(1432)(AA)_2(1234)(1342)$

A2 : $(BB)_1(1342)(1423)(BB)_2(1243)(1423)$

A2 : $(BB)_1(1243)(1423)(BB)_2(1342)(1423)$

A2 : $(BB)_1(1423)(1243)(BB)_2(1324)(1243)$

A2 : $(BB)_1(1324)(1243)(BB)_2(1423)(1243)$

A2 : $(BB)_1(1342)(1423)(BB)_2(1423)(1243)$

A2 : $(BB)_1(1243)(1423)(BB)_2(1324)(1243)$

A2 : $(BB)_1(1423)(1243)(BB)_2(1342)(1423)$

A2 : $(BB)_1(1324)(1243)(BB)_2(1243)(1423)$

A1 : $(CC)_1(1432)(1324)(CC)_2(1234)(1324)$

A1 : $(CC)_1(1234)(1324)(CC)_2(1432)(1324)$

A1 : $(CC)_1(1324)(1234)(CC)_2(1423)(1234)$

A1 : $(CC)_1(1423)(1234)(CC)_2(1324)(1234)$

A1 : $(CC)_1(1324)(1234)(CC)_2(1432)(1324)$

A1 : $(CC)_1(1423)(1234)(CC)_2(1234)(1324)$

A1 : $(CC)_1(1432)(1324)(CC)_2(1324)(1234)$

A1 : $(CC)_1(1234)(1324)(CC)_2(1423)(1234)$

Appendix 5.5 (see subsection 5.5.1c). The values of EUU_{16}, EUV_{16}, EVU_{16} and EVV_{16} in designs for 2×2 treatments per period.

Designs with $EUU_{16} = EUV_{16} = EVU_{16} = 0.982$ and $EVV_{16} = 0.839$

DD(1423)(1324)

DD(1324)(1423)

DD(1423)(1423)

DA(1423)(1324)

Appendix 5.5 (continued)

DA(1423)(1423)

AD(1324)(1423)

AD(1423)(1423)

A1 : $(DD)_1$(1324)(1423)$(AD)_2$(1324)(1423)

A1 : $(DD)_1$(1423)(1423)$(AD)_2$(1423)(1423)

A1 : $(DD)_1$(1324)(1423)$(AD)_2$(1423)(1423)

A1 : $(DD)_1$(1423)(1423)$(AD)_2$(1324)(1423)

A1 : $(AD)_1$(1324)(1423)$(DD)_2$(1324)(1423)

A1 : $(AD)_1$(1423)(1423)$(DD)_2$(1423)(1423)

A1 : $(AD)_1$(1423)(1423)$(DD)_2$(1324)(1423)

A1 : $(AD)_1$(1324)(1423)$(DD)_2$(1423)(1423)

A1 : $(DD)_1$(1324)(1423)$(DD)_2$(1423)(1423)

A1 : $(DD)_1$(1423)(1423)$(DD)_2$(1324)(1423)

A1 : $(AD)_1$(1324)(1423)$(AD)_2$(1423)(1423)

A1 : $(AD)_1$(1423)(1423)$(AD)_2$(1324)(1423)

A2 : $(DD)_1$(1324)(1423)$(AD)_2$(1324)(1423)

A2 : $(DD)_1$(1423)(1423)$(AD)_2$(1423)(1423)

A2 : $(DD)_1$(1324)(1423)$(AD)_2$(1423)(1423)

A2 : $(DD)_1$(1423)(1423)$(AD)_2$(1324)(1423)

A2 : $(AD)_1$(1324)(1423)$(DD)_2$(1324)(1423)

A2 : $(AD)_1$(1423)(1423)$(DD)_2$(1423)(1423)

A2 : $(AD)_1$(1423)(1423)$(DD)_2$(1324)(1423)

A2 : $(AD)_1$(1324)(1423)$(DD)_2$(1423)(1423)

A2 : $(DD)_1$(1324)(1423)$(DD)_2$(1423)(1423)

A2 : $(DD)_1$(1423)(1423)$(DD)_2$(1324)(1423)

A2 : $(AD)_1$(1324)(1423)$(AD)_2$(1423)(1423)

A2 : $(AD)_1$(1423)(1423)$(AD)_2$(1324)(1423)

Appendix 5.5 (continued)

A3 : $(DD)_1(1423)(1324)(AA)_2(1423)(1324)$

A3 : $(DD)_1(1423)(1324)(AA)_2(1324)(1324)$

A3 : $(DD)_1(1324)(1423)(AA)_2(1324)(1324)$

A3 : $(DD)_1(1423)(1423)(AA)_2(1423)(1324)$

A3 : $(DD)_1(1324)(1423)(AA)_2(1423)(1324)$

A3 : $(DD)_1(1423)(1423)(AA)_2(1324)(1324)$

A3 : $(AA)_1(1324)(1423)(DD)_2(1324)(1423)$

A3 : $(AA)_1(1423)(1423)(DD)_2(1423)(1423)$

A3 : $(AA)_1(1423)(1423)(DD)_2(1324)(1423)$

A3 : $(AA)_1(1324)(1423)(DD)_2(1423)(1423)$

A3 : $(AA)_1(1423)(1423)(DD)_2(1423)(1324)$

A3 : $(AA)_1(1324)(1423)(DD)_2(1423)(1324)$

A3 : $(DD)_1(1324)(1423)(DD)_2(1423)(1423)$

A3 : $(DD)_1(1423)(1423)(DD)_2(1324)(1423)$

A3 : $(DD)_1(1423)(1324)(DD)_2(1423)(1423)$

A3 : $(DD)_1(1423)(1423)(DD)_2(1423)(1324)$

A3 : $(DD)_1(1423)(1324)(DD)_2(1324)(1423)$

A3 : $(DD)_1(1324)(1423)(DD)_2(1423)(1324)$

A3 : $(AA)_1(1324)(1423)(AA)_2(1324)(1324)$

A3 : $(AA)_1(1423)(1423)(AA)_2(1423)(1324)$

A3 : $(AA)_1(1324)(1423)(AA)_2(1423)(1324)$

A3 : $(AA)_1(1423)(1423)(AA)_2(1324)(1324)$

Designs with $EUU_{16} = 0.839$, $EUV_{16} = EVU_{16} = EVV_{16} = 0.982$.

DD(1432)(1234)

DD(1432)(1432)

DD(1234)(1432)

DB(1432)(1234)

DB(1432)(1432)

Appendix 5.5 (continued)

BD(1432)(1432)

BD(1234)(1432)

A1 : $(DD)_1(1432)(1432)(BD)_2(1432)(1432)$

A1 : $(DD)_1(1234)(1432)(BD)_2(1234)(1432)$

A1 : $(DD)_1(1432)(1432)(BD)_2(1234)(1432)$

A1 : $(DD)_1(1234)(1432)(BD)_2(1432)(1432)$

A1 : $(BD)_1(1432)(1432)(DD)_2(1432)(1432)$

A1 : $(BD)_1(1234)(1432)(DD)_2(1234)(1432)$

A1 : $(BD)_1(1234)(1432)(DD)_2(1432)(1432)$

A1 : $(BD)_1(1432)(1432)(DD)_2(1234)(1432)$

A1 : $(DD)_1(1432)(1432)(DD)_2(1234)(1432)$

A1 : $(DD)_1(1234)(1432)(DD)_2(1432)(1432)$

A1 : $(BD)_1(1432)(1432)(BD)_2(1234)(1432)$

A1 : $(BD)_1(1234)(1432)(BD)_2(1432)(1432)$

A2 : $(DD)_1(1432)(1432)(BB)_2(1234)(1234)$

A2 : $(DD)_1(1234)(1432)(BB)_2(1432)(1234)$

A2 : $(BB)_1(1432)(1432)(DD)_2(1432)(1432)$

A2 : $(BB)_1(1234)(1432)(DD)_2(1234)(1432)$

A2 : $(BB)_1(1234)(1432)(DD)_2(1432)(1432)$

A2 : $(BB)_1(1432)(1432)(DD)_2(1234)(1432)$

A2 : $(BB)_1(1432)(1432)(DD)_2(1432)(1234)$

A2 : $(BB)_1(1234)(1432)(DD)_2(1432)(1234)$

A2 : $(DD)_1(1432)(1432)(DD)_2(1234)(1432)$

A2 : $(DD)_1(1234)(1432)(DD)_2(1432)(1432)$

A2 : $(DD)_1(1432)(1432)(DD)_2(1432)(1234)$

A2 : $(DD)_1(1432)(1234)(DD)_2(1234)(1432)$

A2 : $(DD)_1(1234)(1432)(DD)_2(1432)(1234)$

A2 : $(BB)_1(1432)(1432)(BB)_2(1432)(1234)$

Appendix 5.5 (continued)

A2 : $(BB)_1(1234)(1432)(BB)_2(1234)(1234)$

A2 : $(BB)_1(1432)(1432)(BB)_2(1234)(1234)$

A2 : $(BB)_1(1234)(1432)(BB)_2(1432)(1234)$

A3 : $(DD)_1(1432)(1432)(BD)_2(1432)(1432)$

A3 : $(DD)_1(1234)(1432)(BD)_2(1234)(1432)$

A3 : $(DD)_1(1432)(1432)(BD)_2(1234)(1432)$

A3 : $(DD)_1(1234)(1432)(BD)_2(1432)(1432)$

A3 : $(BD)_1(1432)(1432)(DD)_2(1432)(1432)$

A3 : $(BD)_1(1234)(1432)(DD)_2(1234)(1432)$

A3 : $(BD)_1(1234)(1432)(DD)_2(1423)(1432)$

A3 : $(BD)_1(1432)(1432)(DD)_2(1234)(1432)$

A3 : $(DD)_1(1432)(1432)(DD)_2(1234)(1432)$

A3 : $(DD)_1(1234)(1432)(DD)_2(1432)(1432)$

A3 : $(BD)_1(1432)(1432)(BD)_2(1234)(1432)$

A3 : $(BD)_1(1234)(1432)(BD)_2(1432)(1432)$

A2 : $(DD)_1(1432)(1234)(BB)_2(1432)(1234)$

A2 : $(DD)_1(1432)(1234)(BB)_2(1234)(1234)$

A2 : $(DD)_1(1432)(1432)(BB)_2(1432)(1234)$

A2 : $(DD)_1(1234)(1432)(BB)_2(1234)(1234)$

Designs with $EUU_{16} = 0.982$, $EUV_{16} = EVU_{16} = 0.946$, $EVV_{16} = 0.911$.

A1 : $(CA)_1(1432)(1324)(CA)_2(1423)(1324)$

A1 : $(CA)_1(1432)(1423)(CA)_2(1423)(1423)$

Designs with $EUU_{16} = 0.911$, $EUV_{16} = EVU_{16} = 0.946$, $EVV_{16} = 0.982$.

A1 : $(CB)_1(1423)(1234)(CB)_2(1432)(1234)$

A1 : $(CB)_1(1423)(1432)(CB)_2(1432)(1432)$

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Appendix A

Some matrix definitions

1. $1_{(n)}$ is an $(n \times 1)$ column vector with all elements unity.

2. $\underset{(m \times n)}{J} = 1_{(m)} 1^T_{(n)}$. In particular $J_{(n)} = 1_{(n)} 1^T_{(n)}$.

3. $I_{(n)}$ is an $(n \times n)$ identity matrix with

$$\text{cell } (i, j) = 1 , \qquad i = j ;$$
$$= 0 , \qquad i \neq j .$$

4. An $(n \times n)$ matrix A is an idempotent matrix if $A \cdot A = A$.

5. A permutation matrix A is an $(n \times n)$ matrix with cell $(i, j) = a_{ij}$ where a_{ij}'s are either 0 or 1

and $\sum_{i=1}^{n} a_{ij} = 1$ for all j .

$\sum_{j=1}^{n} a_{ij} = 1$ for all i .

Appendix B

Direct Product.

Let A be a $J \times K$ matrix with cell $(j, k) = a_{jk}$ and B be a $L \times M$ matrix with cell $(l, m) = b_{lm}$. Then $A \otimes B$ is a $(JL) \times (KM)$ block matrix given by

$$\begin{bmatrix} a_{11} B & \cdots & a_{1K} B \\ \vdots & & \vdots \\ a_{J1} B & \cdots & a_{JK} B \end{bmatrix} = C$$

Let cell (r, s) of $C = c_{rs}$.

Then $c_{rs} = a_{jk} b_{lm}$

where $r - 1 = L(j-1) + l - 1$

$s - 1 = M(k-1) + m - 1$.

The following theorems on direct product, given below without proof, are useful for this thesis.

B.1. $(A \otimes B) \otimes C = A \otimes (B \otimes C)$

B.2. $(A_1 + A_2) \otimes B = A_1 \otimes B + A_2 \otimes B$ where matrices A_1, A_2 are of the same size.

B.3. $(A \otimes B)^T = A^T \otimes B^T$

B.4. $\mathrm{tr}\,(A \otimes B) = (\mathrm{tr}\,A)(\mathrm{tr}\,B)$, tr denotes trace.

B.5. $(A_1 \otimes B_1)(A_2 \otimes B_2) = (A_1 A_2) \otimes (B_1 B_2)$, provided the matrices A_1, A_2, B_1, B_2 have appropriate dimensions.

Appendix C

Matrix Theory

Let A be a square $J \times J$ matrix. $\mathrm{Det}(A - \lambda I_{(J)})$ is a polynomial of order J in λ. It will have J (complex-valued) roots $\lambda_j = \lambda_j(A)$, $j = 1, \ldots, J$. These are called the eigenvalues of A.

As $A - \lambda_j I_{(J)}$ is singular, there is a $J \times 1$ vector Z_j, not all of whose elements are zero, such that

$$(A - \lambda_j I_{(J)}) Z_j = 0$$

Z_j is called an eigenvector corresponding to λ_j.

The following theorems, given without proof, hold for A a symmetric matrix. They are useful for section 4 of Chapter Five.

Theorem C.1. All the eigenvalues are real and the eigenvalues can be chosen to be real.

Theorem C.2. The eigenvectors corresponding to distinct eigenvalues are orthogonal.

Theorem C.3. There exist mutually orthonormal eigenvectors $Z_1, \ldots, Z_J$ such that

$$A Z_j = \lambda_j Z_j$$

that is, $$A Z = Z D$$

where $Z = [Z_1 \ldots Z_J]$ is orthonormal

and $D = \mathrm{diag}\left\{\lambda_j,\ j = 1, \ldots, J\right\}$

Corollary:

$$Z^T A Z = D$$

$$A = Z D Z^T \quad .$$

Therefore,

$$A = \lambda_1 Z_1 Z_1^T + \ldots + \lambda_J Z_J Z_J^T$$

$$I = Z_1 Z_1^T + \ldots + Z_J Z_J^T \quad .$$

Theorem C.4. rank (A) = number of non-zero λ_j (A)

$$\text{trace}\,(A) = \sum_{j=1}^{J} \lambda_j\,(A)$$

$$\text{Det}\,(A) = \prod_{j=1}^{J} \lambda_j\,(A)$$

Theorem C.5. If $\text{Det}(A) \neq 0$, then

$$A^{-1} = Z\,D^{-1}\,Z^T$$

$$\lambda_j\,(A^{-1}) = [\lambda_j\,(A)]^{-1}$$

$$Z_j\,(A^{-1}) = Z_j\,(A) \quad .$$

Theorem C.6. (Invariance property).

If P is a $J \times J$ orthogonal matrix such that $P\,P^T = I_{(J)}$,

then

$$\lambda_j\,(P\,A\,P^T) = \lambda_j\,(A)$$

$$Z_j\,(P\,A\,P^T) = P^T\,Z_j\,(A) \quad .$$